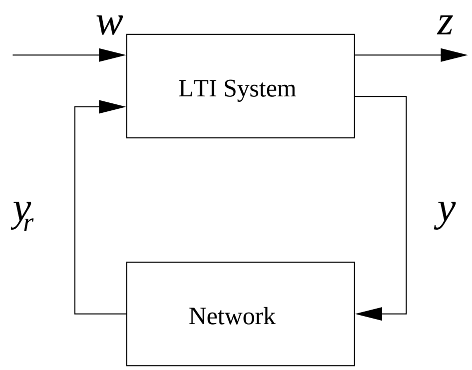
EE 150 Presentation

Week 6

L. Xiao, M. Johansson, H. Hindi, S. Boyd, A. Goldsmith: "Joint Optimization of Communication Rates and Linear Systems"

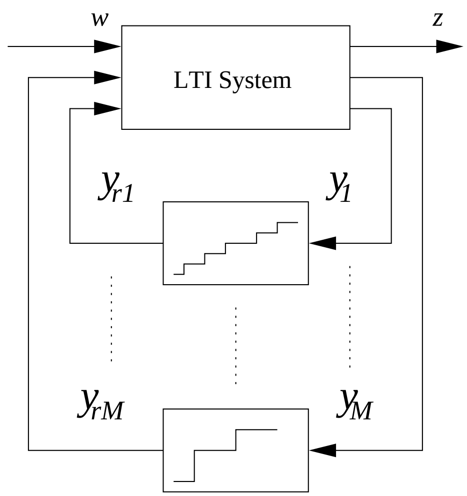
Ather Gattami, May 6, 2003

Problem Setup



- w: exogenous signal, including noises, disturbances, and commands.
- z: output signal that contains the critical performance variables.
- y: continuous signal transmitted over the network.
- y_r: recieved decoded version of y.

Problem Setup



- The signals y_i are coded using memoryless uniform quantizers, with possibly different numbers of bits and scale factors.
- Time-delays assumed known and fixed, and modeled as part of the LTI system.

Problem Setup

Many issues arise in the design of networked controllers:

- Bit rate limitations.
- Unmodeled time-delays.
- Packet loss.
- Transmission errors.
- Asynchronicity.

We will only consider the first issue, where the other ones assumed not to occur.

Problem Setup

The paper concentrates on certain critical communication parameters such as:

- Individual channel transmission powers.
- Bandwiths allocated to the channels(or groups of channels).
- Time-slot fractions allocated to the channels(or group of channels).

We will refer to these parameters collectively as *communication variables*.

The communication variables indirectly limit the number of bits allocated to each quantizer.

Linear System Model

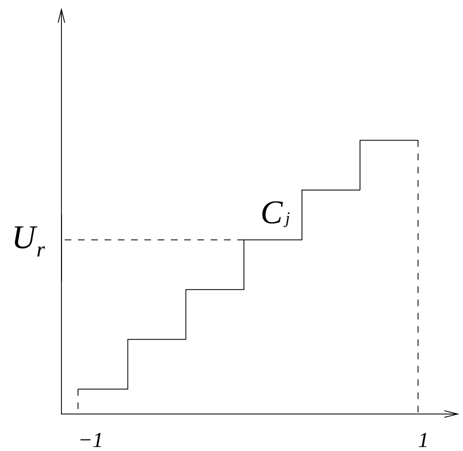
The LTI is described by

$$z = G_{11}(\phi)w + G_{12}(\phi)y_r$$

$$y = G_{21}(\phi)w + G_{22}(\phi)y_r$$

- G_{ij} are LTI operators.
- $\phi \in \mathcal{R}^q$ is the vector of design parameters in the LTI system that can be tuned or changed to optimize the system.
- $y(t), y_r(t) \in \mathcal{R}^M$ are transmitted over the network during each sampling period.
- all communcation delays are assumed constant and known, and included in the LTI system model.

Quantization Model



A unit range uniform b-bit quantizer partitions the range $[-1, 1]$ into 2^b intervals of unifrom width 2^{1-b}.

- Each quantization interval is assigned a codeword of b bits.
- Given the associated codeword, the value is reconstructed as u_r, which is the midpoint of the interval corresponding to the codeword.

Quantization Model

- The relation between the original and reconstructed values is given by

$$\frac{\mathbf{round}(2^{b-1})}{2^{b-1}}$$

for $|u| \leq 1$(which means an assumption of no overflow).

$\mathbf{round}(z)$ is the integer nearest to z:

$$\mathbf{round}(2.73) = 3,$$

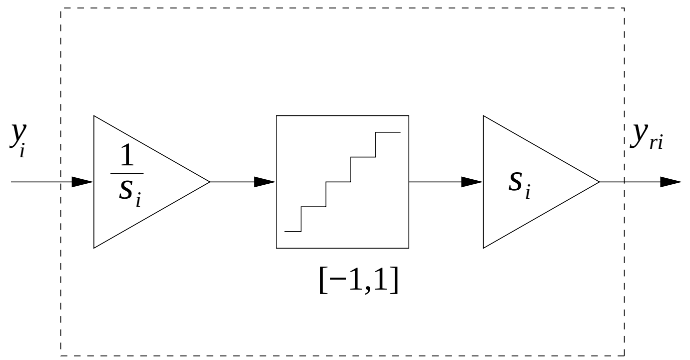
$$\mathbf{round}(-5.3) = -5.$$

- The associated quantization error is given by

$$E_b(u) = u_r - u = \frac{\mathbf{round}(2^{b-1}u) - 2^{b-1}u}{2^{b-1}}$$

- $|\mathbf{round}(2^{b-1}u) - 2^{b-1}u| \leq 2^{-1} \Longrightarrow |E_b(u)| \leq 2^{-b}$.

Scaling



- To avoid overflow, each $y_i(t)$ is scaled by the factor $s_i^{-1} > 0$ prior to encoding and again rescaled by the factor s_i, that is

$$y_{ri}(t) = s_i Q_{b_i}(y_i(t)/s_i).$$

- Assuming $|y_i(t)| < s_i$, the associated quantization error is given by

$$|q_i(t)| = |y_{ri}(t) - y_i(t)| = |s_i E_{b_i}(y_i(t)/s_i)| < s_i 2^{-b_i}.$$

Scaling

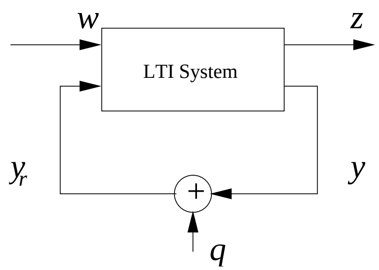
- Possible choice for s_i is the maximum possible value of $|y_i(t)|$, or a value with very high probability larger than $|y_i(t)|$.

  Example: If y_i is a Gaussian amplitude distribution, the choice (by the so-called 3σ*-rule*)

  $$s_i = 3\mathbf{rms}(y_i)$$

  ensures overlow occures only about 0.3% of the time.

Quantization Error Model



- Model the quantization errors $q_i(t)$ as white noise, uniformly distributed on the interval

$$s_i[-2^{-b_i}, 2^{-b_i}].$$

with zero mean and variance

$$\mathbf{E} q_i(t)^2 = (1/3) s_i^2 2^{-2b_i}.$$

Performance

- Write z and y in terms of the inputs w and q:

$$z = G_{zw} w + G_{zq} q, \quad y = G_{yw} w + G_{yq} q,$$

where

$$\begin{aligned} G_{zw} &= G_{11} + G_{12}(I - G_{22})^{-1} G_{21}, & G_{zw} &= G_{12} + G_{21}(I - G_{22})^{-1} G_{22} \\ G_{zw} &= (I - G_{22})^{-1} G_{21}, & G_{zw} &= (I - G_{22})^{-1} G_{22} \end{aligned}$$

are the closed-loop transfer matrices from w and q to z and y, resp.

- The variance z introduced by the quantization is given by

$$V_q = \mathbf{E}||G_{zq}||^2 = \sum_{i=1}^{M} ||G_{zqi}||^2 (1/3) s_i^2 2^{-2b_i},$$

where $||\cdot||$ denotes the $\mathbf{L}^2$ norm.

Performance

- V_q is used as a measure of the effect of quantization on the overall system performance

  – Assume w is modeled as a stationary stochastic process. Then the overall variance of z is given by

$$V = \mathbf{E}||z||^2 = V_q + \mathbf{E}||G_{zw}w||^2.$$

- let

$$V_q = \sum_{i=1}^{M} a_i 2^{-2b_i}$$

  where $a_i = ||G_{zqi}||^2 (1/3) s_i^2$.

- The expression shows how V_q depends on the allocation of quantizer bits $b_1, \ldots, b_M$, as well as the scalings $s_1, \ldots, s_M$.

Communications Model

- Let θ denote the vector of communication variables.
- Let $b \in \mathbf{R}^M$ denote the vector of bits allocated to each quantized signal.
- The associated communication rate r_i(bits/second) can be expressed as $b_i = \alpha r_i$, where

  $\alpha = c_s / f_s$,

  $f_s =$ sample frequency,

  $c_s =$ channel coding efficiency(source bits/transmission bit).
- Hence capacity constraitns expressed in terms of bit allocations.

Communications Model

General model to relate the vector of bit allocations b and the vector of communication variables θ:

$$\begin{aligned} f_i(b,\theta) &\leq 0, \quad i = 1, ..., m_f \\ h_i^T\theta &\leq d_i, \quad i = 1, ..., m_h \\ \theta_i &\geq 0, \quad i = 1, ..., m_\theta \\ \underline{b}_i \leq b_i &\leq \bar{b}_i, \quad i = 1, ..., M \end{aligned}$$

Model Assumptions

- f_i are convex functions of (b, θ), monotone increasing in b and monotone decreasing in θ. Roughly speaking, the conditions mean that the capacity of the channels increase with increasing resources.
- Second set of constraints describe resource limitations. Assume $h_i \geq 0$, $d_i > 0$.
- Communication resource variables assumed to be nonnegative.
- Lower and upper bounds for each bit allocation is assumed to be nonnegative. Lower bounds are imposed to ensure that the white noise model for quantization errors is reasonable. Upper bounds arise from hardware limitations.

Examples of Channel Capacity Contraints

Frequency Division Multiple Access (FDMA) Gaussian Channels:

- A transmitter sends information to n recievers over disoint frequency bands with bandwiths $W_i \geq 0$ and assigns a transmit power $P_i \geq 0$ to each band.
- Recievers subject to independent additive white Gaussian noises with power spectral densities N_i.

Examples of Channel Capacity Contraints

By the so-called Shannon capacity result, bit allocations b_i and communications variables $\theta_i = (P_i, W_i)$ are related by

$$b_i \leq \alpha W_i \log_2(1 + (P_i/N_i W_i))$$

$$\Updownarrow$$

$$f_i(b_i, W_i, P_i) = b_i - \alpha W_i \log_2(1 + (P_i/N_i W_i)) \leq 0,$$

$$i = 1, ..., n.$$

Fits our generic form!

Examples of Channel Capacity Contraints

- $f_i(b_i, W_i, P_i)$ is increasing in b_i, decreasing in W_i and P_i.
- The function $g(P) = -\alpha \log_2(1 + (P/N)$ is convex(easly verified). Therefore its *perspective function*

$$Wg(P/W) = -W\alpha \log_2(1 + (P/WN)$$

is also convex. Consult (Boyd and Vanderberghe, Dec 2002) for a proof. Hence f is convex.

- The communication variables are constrained by total resource limit

$$P_1 + ...P_n \leq P_{tot}$$

$$W_1 + ...W_n \leq W_{tot}$$

which have the generic form for total resource limit.

Resource Allocation for Fixed Linear System

Assume Linear System is fixed.

- Objective: minimize variance of the performance signal z.
- Problem can be formulated as convex optimization problem:

$$\text{minimize} \sum_{i}^{M} a_i 2^{-2b_i}$$

subject to

$$\begin{aligned}
f_i(b, \theta) &\leq 0, \quad i = 1, ..., m_f \\
h_i^T \theta &\leq d_i, \quad i = 1, ..., m_h \\
\theta_i &\geq 0, \quad i = 1, ..., m_\theta \\
\underline{b}_i \leq b_i &\leq \bar{b}_i, \quad i = 1, ..., M
\end{aligned}$$

Joint Design of Communication and Linear Systems

- To optimize system performance, optimization of the parameters of the linear system *and* the communication system should be done *jointly*.
- Joint design problem is in general not convex!

Alternating Optimization for Joint Design

- Use heurestic method:

  given initial linear system variables ϕ_0, communication variables θ_0, scalings s_0

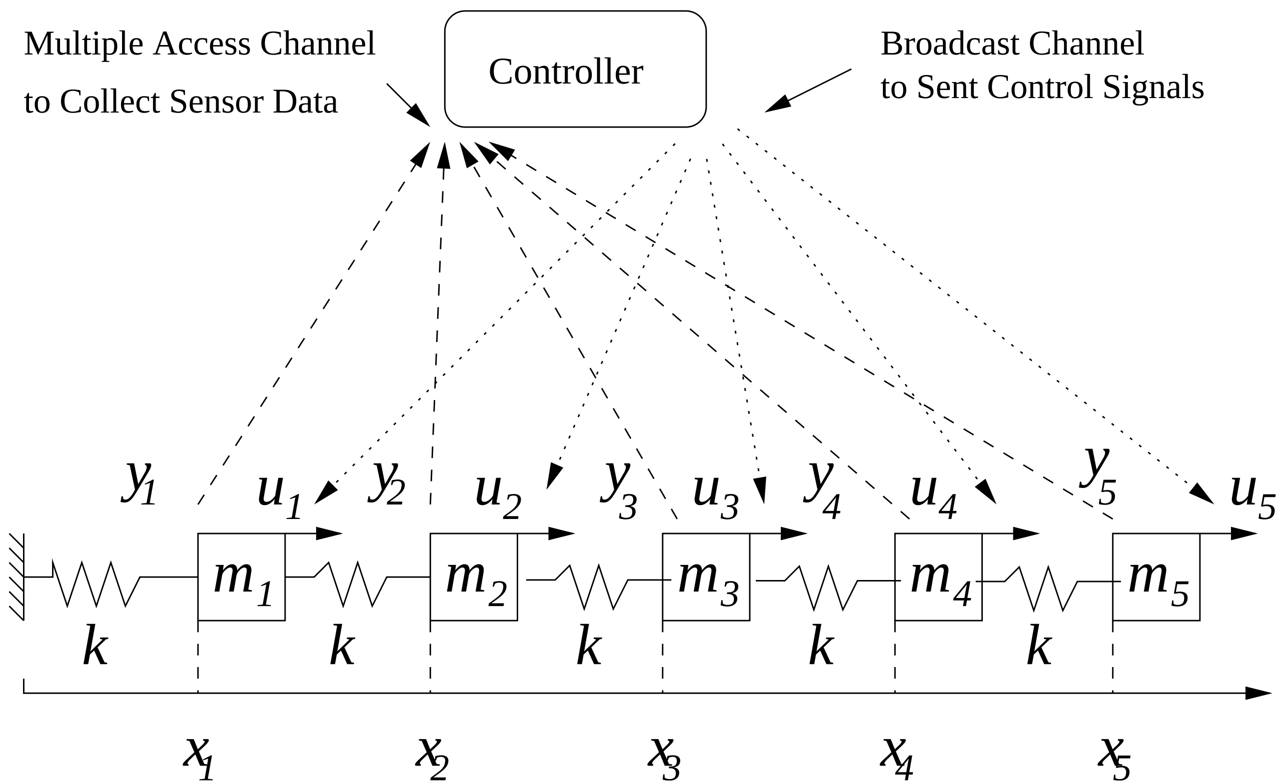
  repeat

  1. Fix ϕ_k, s_k and optimize over θ. Let θ_{k+1} be the optimal value.
  2. Fix θ_{k+1}, s_k and optimize over ϕ. Let ϕ_{k+1} be the optimal value.
  3. Fix ϕ_{k+1}, θ_{k+1}. Let s_{k+1} be appropriate scaling factors.

  until convergence.

- Because of nonconvexity of the joint problem, convergence is *not* guarenteed.
- Works well in practice.

Numerical example: control of mass-spring system



Numerical example: control of mass-spring system

- Position sensors send measurements $y_i = x_i + v_i$, where v_i is the sensor noise, to the controller through a Gaussian multiple access using FDMA.
- The controller recieves $y_i = x_i + v_i + q_i$, where q_i is the quantization error.
- Actual force acting on each mass is $u_{rj} = u_j + w_j + p_j$, where w_j is the exogenous disturbance and p_j is the quantization error.
- Mechanical system parameters: $m_1 = 10, m_2 = 5, m_3 = 20, m_4 = 2,$ $m_1 = 15$, and $k = 1$.
- Discrete-time system is obtained using a sampling frequency that is 5 times faster than the fastest mode of the coninuous-time dynamics.

Numerical example: control of mass-spring system

- w and v independent zero-mean white noises with covariance matrices

$$R_w = R_v = 10^{-6} I.$$

- Actuators impose rms constraints on the control signals:

$$\mathbf{rms}(u_i) \leq 1, \quad i = 1, ..., 5$$

- The multiple access channel(mac) and broadcast channel(bc) have separate total power limits $P_{\text{mac,tot}} = P_{\text{bc,tot}} = 7.5$, but they share a total bandwidth limit $W_{\text{tot}} = 10$.

- All recievers have the same noise power density $N = 0.1$.

- Impose an upperbound $\bar{b} = 12$ and a lower bound $\underline{b} = 5$ for all quantizers.

Numerical example: control of mass-spring system

- Allocate power and bandwidth evenly to all sensors, which results in a uniform allocation of eight bits for each channel.
- We want to minimize $\mathbf{rms}(z)$.
  - Found $\mathbf{rms}(z) = 0.549$ with fixed resource allocation.
  - Using the alternating procedure, to do joint optimization of bit allocation and controller design, we get $\mathbf{rms}(z) = 0.116$ after four iterations. Hence, a reduction of 77% in rms value compared with uniform bit allocation.