EE 150 Presentation

Week 5

David F. Delchamps: “Extracting state information from a quantized output record” (1989), and “Stabilizing a Linear System With Quantized State Feedback” (1990)

Lars B. Cremean, April 29, 2003

Motivation

- We wish to control linear, time-invariant systems (P). Typically, $x(\cdot) \in \mathbb{R}^n$, $u(\cdot) \in \mathbb{R}^m$, $y(\cdot) \in \mathbb{R}^p$. Unity feedback block diagrams shown, with controller C.

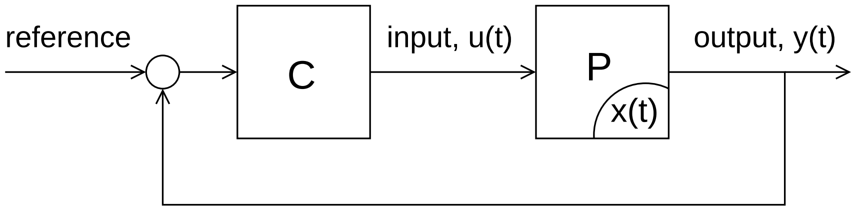

Figure 1: Continuous-time, $t \in \mathbb{R}$.

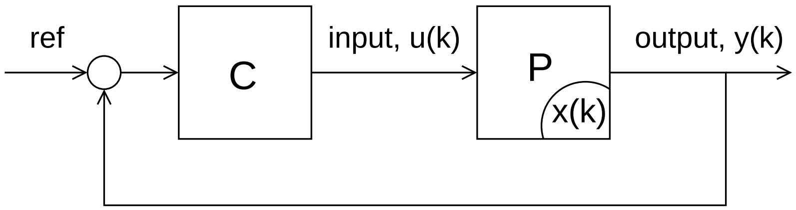

Figure 2: Discrete-time, $k \in \mathbb{Z}$.

Motivation (2)

- In *quantized output* systems, the output $y(\cdot)$ is restricted to a *discrete*, countable set of values, e.g. $y(k) \in \mathbf{Y} \subset \mathbb{R}^p$ where $\mathbf{Y} = \{y_i : i \in \mathbb{Z}\}$.
- In these papers, the output is taken to be a quantization of the state, i.e. the state passes through a *quantizer* $q(\cdot)$ to get to the output.

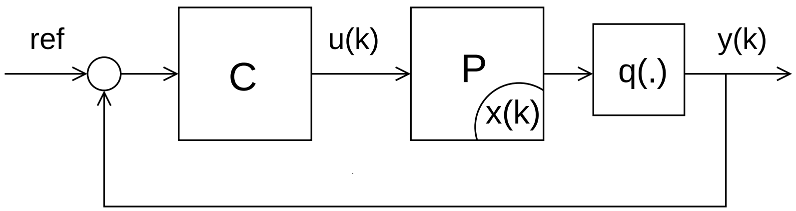

Figure 3: Discrete-time system with quantized output.

Motivation (3)

- Most modern control applications are implemented digitally.
- Quantization is an **inherent** part of digital implementation.

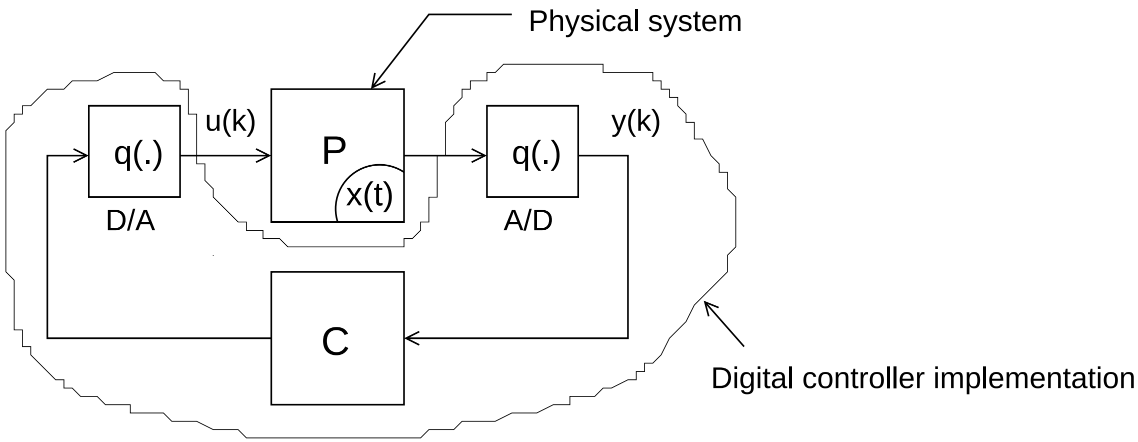

Figure 4: Typical controller implementation methodology.

Paper #1: Extracting state information

"Extracting state information from a quantized output record", Systems & Control Letters 13 (1989) 365-372

- **Underlying motivation**: Implementations of control technology becoming overwhelmingly digital rather than analog. Want to study interface between continuous systems and discrete implementation.
- Quantization often regarded as approximation of continuous measured state.
- Correlation of successive quantization errors usually not considered.
- **Idea**: Take quantized output *record* of a system to obtain a "better" determination of the system's state.

Paper #1: Questions

- How much information about the current state of a given system is contained in a long record of past quantized measurements of the system's output?
- How can we manipulate the system's input to make the output record more informative about the state evolution?

Form of questions reminiscent of idea of *observability* in control theory.

Domain of consideration

- Discrete-time, linear, time-invariant (LTI) state-space systems

$$x(k+1) = ax(k) + bu(k), \quad k \geq 0 \tag{1a}$$

$$y(k) = q(x(k)) \tag{1b}$$

- In general, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$
- In this paper, consideration restricted to case where the state dimension is one ($n = 1$) and the single-input, single-output (SISO) case ($m = 1$, $p = 1$): $x(\cdot) \in \mathbb{R},\ a, b \in \mathbb{R},\ u(\cdot), y(\cdot) \in \mathbb{R}$

Quantization

- $q(\cdot)$ is a mapping from state space ($\mathbb{R}$) into a countable set $\mathbf{Y} = \{y_i : i \in \mathbf{Z}\}$
- q induces a partition on the state space, $\{U_i\}$, where $q(x) = y_i$, $\forall x \in U_i$.
- *Uniform quantizer with sensitivity Δ:*

$$q_\Delta(x) = \begin{cases} i & \text{when } x \geq 0 \text{ and } x \in [(i - \frac{1}{2})\Delta, (i + \frac{1}{2})\Delta) \\ -q_\Delta(-x) & \text{when } x < 0 \end{cases}$$

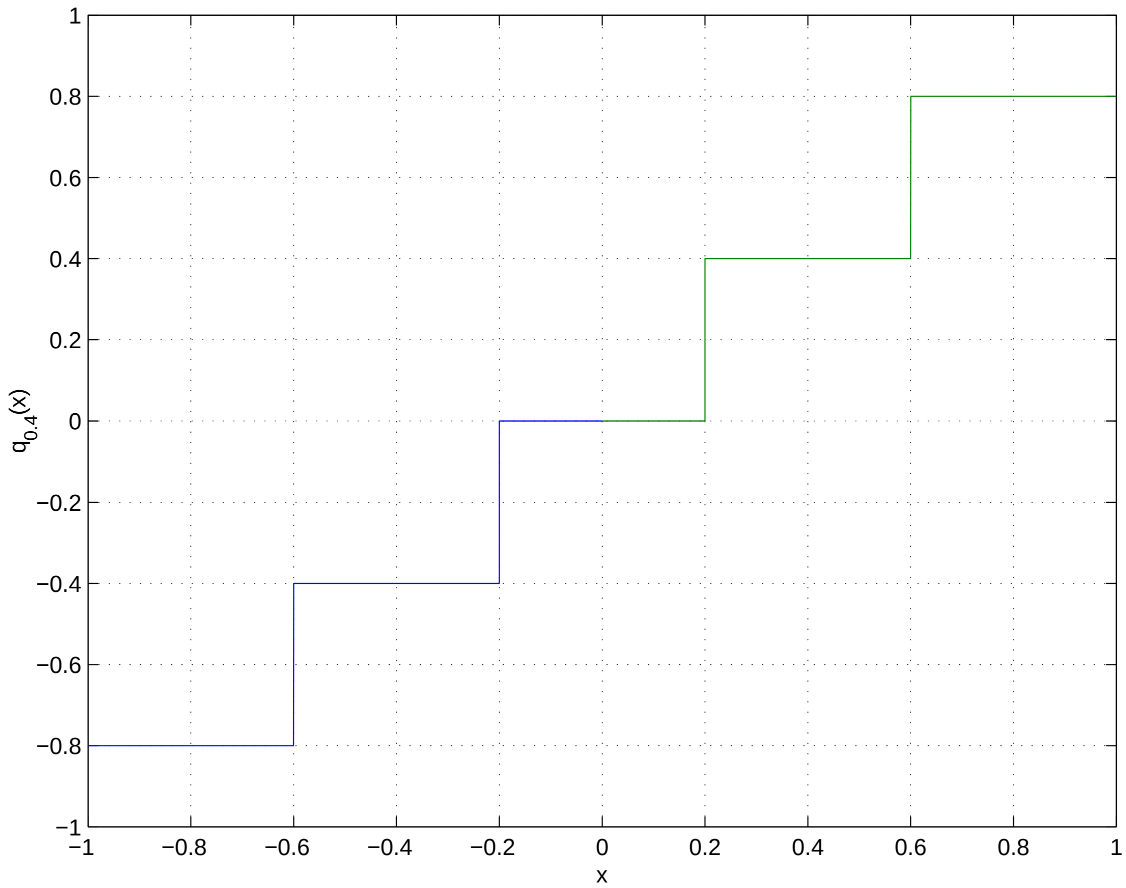

Figure 5: Uniform quantizer with sensitivity $\Delta = 0.4$.

Output record

- Recall the output equation $y(k) = q(x(k))$
- Def ine the *output record* at time k as $\{y(l) = q(x(l)) : 0 \leq l \leq k\}$.
- Def ine *admissible control strategies* as those that depend only on the current output record, i.e.

$$u(k) = f^{(k)}(y(0), \ldots, y(k)), \quad k \geq 0$$

- (Note that this allows time-dependence for controller).

Symbolic dynamics

- Central idea: Assign a symbol to each element of the partition $\{U_i\}$ (e.g. $\{A, B, C, \ldots\}$).
- Using a given fixed control strategy, evolution of state describes a *symbol sequence* associated with its output record.
- *Symbolic dynamics* seeks to relate the **behavior of the closed-loop dynamics** to the **sequence of symbols** that contain information about the trajectory of $x(k)$
- In particular...

Symbolic dynamics: questions

- Given a system of the form $x(k+1) = G(x(k))$,
- To what extent does the symbol sequence associated with $x(0)$ *determine* $x(0)$?
- Under what conditions on the mapping G and the chosen state-space partitioning can we make an “asymptotically perfect” determination of $x(0)$?

Example: Binary shift transformation

- Recall

$$x(k+1) = ax(k) + bu(k), \quad k \geq 0$$
$$y(k) = q(x(k))$$

- Consider the case where $a = 2, b = 1, q(x) = \text{floor}(2x), u = -y$:

$$x(k+1) = 2x(k) + u(k), \quad k \geq 0$$
$$y(k) = \text{floor}(2x(k))$$

Example: Binary shift transformation (2)

- $a = 2, b = 1, q(x) = \text{floor}(2x), u = -y$:

$$x(k+1) = 2x(k) + u(k), \quad k \geq 0$$
$$y(k) = \text{floor}(2x(k))$$

- This map produces the binary expansion of $x(0) \in [0, 1]$, therefore $x(0)$ can be determined to *arbitrary precision*.

- Recall, binary expansion of x is representation $\{a_i\}$ ($a_i \in \{0, 1\}$) such that

$$x = \sum_{i=1}^{\infty} \frac{a_i}{2^i}$$

Feedback regulation of information f low

- In the previous example, information about the *initial condition* is attained by looking at the output record.
- We want to use the same ideas to better estimate the *current* state.

Problem setup

- Assume $x(0)$ is *uniformly distributed* over the quantization interval $[(y_0 - \frac{1}{2})\Delta, (y_0 + \frac{1}{2})\Delta)$.
- Once the control law is chosen, $x(k)$ and $y(k)$ evolve as random processes.
- We wish to minimize our uncertainty about $x(k)$ given the output record $\{y(l) : 0 \leq l \leq k\}$

Preliminaries and notation

- Let y_0^k be shorthand for "$y(0) = y_0$,. . .,$y(k) = y_k$"
- Let $f(x(k)|y_0^k)$ be the conditional probability density of $x(k)$ given y_0^k.
- Def ine the *differential entropy*, or *dispersion* of $x(k)$ as

$$h(x(k)|y_0^k) = -\sum_{(y_0,\ldots,y_k)\in\mathbf{Y}^k} \text{prob}(y_0^k)\left(\int f(x(k)|y_0^k)\log f(x(k)|y_0^k)\,dx(k)\right)$$

- The lower this value, the more information about $x(k)$ is contained in the output record.
- $h(x(0)|y(0)) = \log\Delta$; and, generally, $h(x(k)|y(k)) = \log\Delta$.

Central result

$$\text{Given } x(k+1) = ax(k) + bu(k) \tag{1a}$$

$$y(k) = q(x(k)), \tag{1b}$$

and a family of (admissible) control laws

$$u(k) = f^{(k)}(y(0), \ldots, y(k)), \quad k \geq 0 \tag{2}$$

Theorem (3.1). Assume in (1) that $b \neq 0$.

(a) If $|a| < 2$, then there exists a feedback law of the form (2) that makes $h(x(k)|y_0^k) \to -\infty$ as $k \to \infty$.

(b) If $|a| \geq 2$, then a control law of the form (2) can be chosen so that $h(x(k)|y_0^k)$ approaches a f inite limit smaller than $\log \Delta$.

Proof of central result, part (a)

- Idea: Use control to keep moving $x(k)$ to the junction of two quantization blocks.
- Shown that given this control, $h(x(k)|y_0^k) = k\log(\frac{1}{2}|a|) + \log\Delta$.

Proof of central result, part (b)

- Relies on theory of Markov chains with countable state spaces.
- Idea: Given a control strategy, $x(k)$, given $\{y(l) : l \leq k\}$, is distributed uniformly over an interval of random length $\Lambda(k)$.
- For given control law, $\{\Lambda(k)\}$ is a Markov chain with countable state space L.
- A stationary distribution exists for this Markov chain, and that this limiting distribution results in convergence of $h(x(k)|y_0^k)$ to a value smaller than $\log \Delta$.

Paper #2: Stabilizing a quantized system

"Stabilizing a Linear System with Quantized State Feedback", IEEE Transactions on Automatic Control, Vol. 35, No. 8, August 1990

- **Domain**: (Unstable) discrete-time, linear time-invariant (LTI) system with quantized measurements.
- **Problem statement**: Stabilize the system, that is, find a control law that brings closed-loop trajectories arbitrarily close to the origin for some time.
- Similar idea as paper #1: Consider quantized measurements as partial observations rather than approximations. Use a record of these partial observations to form better state estimates.

Paper #2: Questions

- Under what circumstances and in what sense can we stabilize an unstable LTI system using control based on only past quantized measurements?
- How does the answer to the above question depend on the properties of the system versus properties of the quantizer?

Paper #2: Notation

- Quantization q maps to a countable set J (same as Y in paper #1)
- System equation

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0, \ (x \in \mathbb{R}^n, \ u \in \mathbb{R}^m) \tag{1}$$

- *Admissible* control

$$u(k) = f^{(k)}(q(x_0), q(x(1)), \ldots, q(x(k))) \tag{2}$$

- Let $0 \in \mathbb{R}^n$ lie in the interior of $U_0 = q^{-1}(q(0)) \subset \mathbb{R}^n$, bounded.

Result: No asymptotic stabilization

Proposition 2.1. In (1), suppose that A is unstable. Then for every control law of the form (2), the set of all $x_0 \in \mathbb{R}^n$ whose closed-loop trajectories $k \longrightarrow x(k)$ tend to zero as $k \longrightarrow \infty$ has Lebesgue measure zero.

Stabilization

- In what sense, then (not asymptotically), can we stabilize system (1)?
- From here in, consider only rectilinear uniform quantizers, i.e. given $\Delta_1, \ldots, \Delta_n > 0$,

$$[q_\Delta(x)]_j = q_{\Delta_j}(x_j)$$

  (Partitions $\mathbb{R}^n$ into rectilinear quantization “blocks” whose edges are parallel to co-ordinate axes).
- **Question**: Given $\epsilon > 0$, can we design a control law that get each closed-loop trajectory to get within ϵ of the origin and stay there an arbitrarily long time $K_o > 0$?

Result: Finite-time stabilization to an ϵ-ball

- For system (1), let q be the directionwise uniform quantizer q_Δ as def ined. Let Δ_{max} and Δ_{min} denote the largest and smallest of the $\{\Delta_j\}$

Proposition 2.2. Suppose that $\|A\|_\infty \leq (2\Delta_{min}/\Delta_{max})^{1/n}$ and that (A, B) is controllable. Then for every $\epsilon > 0$ and integer $K_0 > 0$ there exists a $K_1 > 0$ and an admissible control law such that all closed-loop trajectories are within ϵ of the origin for times $k \in [K_1, K_1 + K_o]$.

Result: Naive stabilization "works"

- Design F such that $(A - BF)$ is stable, and choose control law

$$u_i(k) = -F\overline{x}(k), \tag{3}$$

$$\text{where} \quad [\overline{x}(k)]_j = \Delta_j [q_\Delta(x(k))]_j.$$

- $\overline{x}(k)$ describes the center of the quantization block corresponding to $x(k)$.

Proposition 2.3. Let γ be such that $\lambda_{max}(A - BF) < \gamma < 1$ and let Δ_{max} be the largest of the $\{\Delta_j\}$. Then there exists an ellipsoid D centered at $0 \in \mathbb{R}^n$ and $\exists\, N(x_o) > 0$ for which $x(k) \in D$ for all $k \geq N$. If $x_o \in D$, then $N = 0$.

Naive stabilization: analysis

- Dynamics (1) subject to control (3) yield closed-loop system:

$$x(k+1) = G(x(k)) = Ax(k) - BF\overline{x}(k) \tag{5}$$

- Def inining $e(k) \triangleq x(k) - \overline{x}(k)$,

$$x(k+1) = (A - BF)x(k) + BFe(k)$$

Naive stabilization: special case

- Restrict consideration to state dimension $n = 1$:

$$x(k+1) = ax(k) + bu(k), \quad k \geq 0$$
$$(a \in \mathbb{R}, |a| > 1, b \in \mathbb{R}^{1\times m})$$

- Let $f \in \mathbb{R}^m$ be such that $|a - bf| \leq 1$, and (given Δ and uniform quantizer q_Δ) choose control

$$u(k) = -f\Delta q_\Delta(x(k)), \quad k \geq 0$$
$$\Rightarrow x(k+1) = ax(k) - bf\overline{x}(k) \tag{6}$$

Naive stabilization in 1D: invariant sets

- Applying proof of Proposition 2.3, we f ind that the following D is an invariant set of the closed-loop dynamics:

$$D = \left\{ x \in \mathbb{R} : |x| \le \frac{|bf|\,\Delta}{2(1 - |a - bf|)} \right\}$$

- **Example**: $a = 3/2$, $bf = 5/8$, ($a - bf = 7/8$), $D = [-2.5, 2.5]\Delta \subset \mathbb{R}$.
- This is not, however, the smallest invariant set.

Result: smallest symmetric invariant set

- Let $x^* \in \mathbb{R}$ denote $\inf_{x>0} \{x | G([-x, x]) \subset [-x, x]\}$.

Lemma 3.1. Suppose $a > 1$ and $a - bf \in [0, 1)$, then for the closed loop dynamics of (6),

$$x^* = G(N_+ + \tfrac{1}{2}) = [\tfrac{1}{2}a + N_+(a - bf)]\Delta,$$

$$\text{where} \quad N_+ = \min\left\{N \in \mathbb{Z} : N \geq \frac{a - 3 + 2a^{-1}(a - bf)}{2[1 - (a - bf)]}\right\}.$$

- **Example**: $a = 3/2$, $bf = 5/8$, $N_+ = 0$ $(\geq -0.0208) \Rightarrow \; x^* = \frac{3}{4}\Delta$.

Result: D^* is almost globally attracting

$$x(k+1) = ax(k) - bf\overline{x}(k) \tag{6}$$

Lemma 3.2:

- If no fixed points exist outside of $D^* = [-x^*, x^*]$, then every trajectory of (6) enters D^* eventually.
- If fixed points exist outside of D^*, then these are unstable. The set of initial conditions whose trajectories never enter D^* is contained in a set of measure zero.

Result: invariant measures

Theorem 4.3. Assume in (6), suppose that $a \in \mathbb{Z}: \; a \geq 2$ and that $(a - bf)(N_+ + 1) \leq \frac{1}{2}a$. Then

i) there exists on D^* exactly one probability measure μ^* that is both invariant under G and absolutely cts. w.r.t. normalized Lebesgue measure λ on D. The density of μ^* is positive λ-almost everywhere in D^*, and G is ergodic w.r.t. μ^*;

ii) almost every trajectory of (6) is dense in D^*; and

iii) $\lim_{L\to\infty} \frac{1}{L} \sum_{k=0}^{L-1} P_G^k(\phi) = \phi^*$

- ϕ^* is the density of μ^* w.r.t. λ
- $\phi \in L^1(\lambda)$ is the density of an arbitrary probability measure on D^* absolutely continuous w.r.t. λ.

Central themes

- Quantization can be viewed as more than just an approximate measurement.
- Quantization can be taken into account explicitly during control system design, to achieve better closed-loop performance.
- Long-term behavior of quantized systems can be quite different from that expected by treating quantization as a noisy process.

Appendix A. Paper #1 Figures

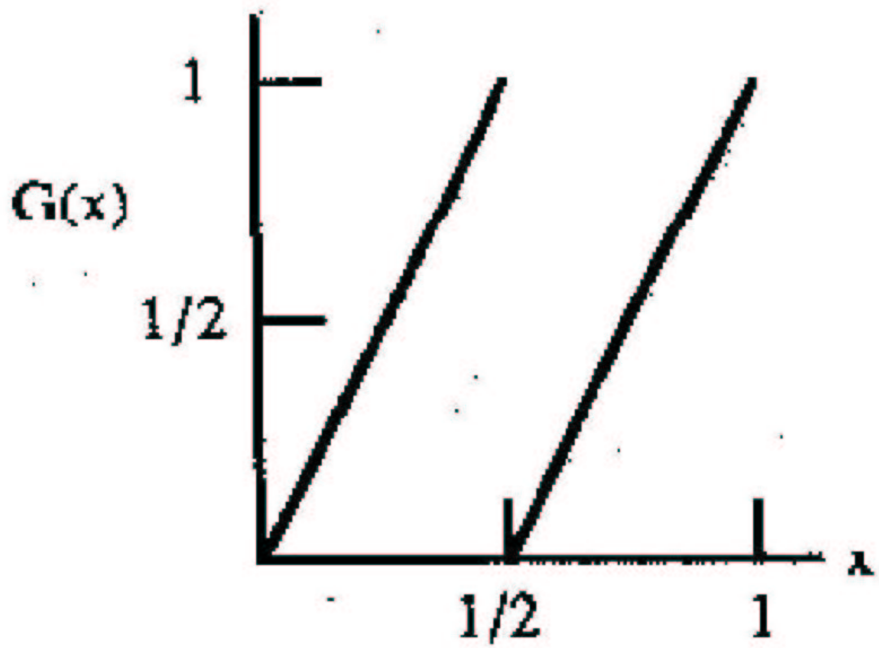

Fig. 1.

Figure 6: Graph of binary shift transformation, $G(x) = \text{mod}(2x, 1)$, $x \in [0, 1]$.

Paper #1 Figures (2)

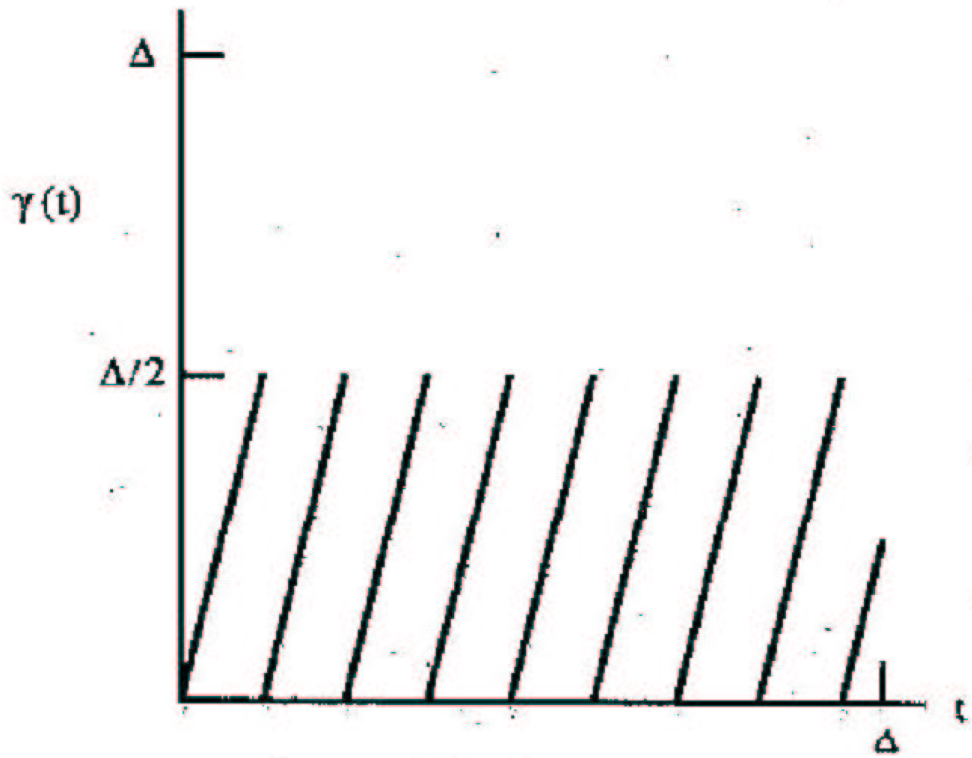

Fig. 2.

Paper #1 Figures (3)

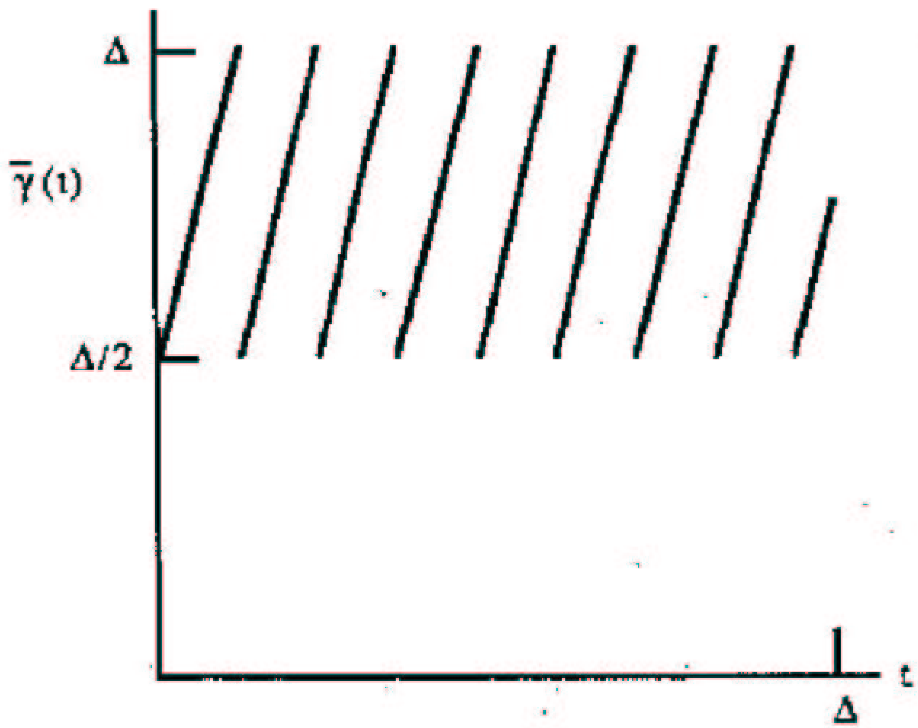

Fig. 3.

Paper #1 Figures (4)

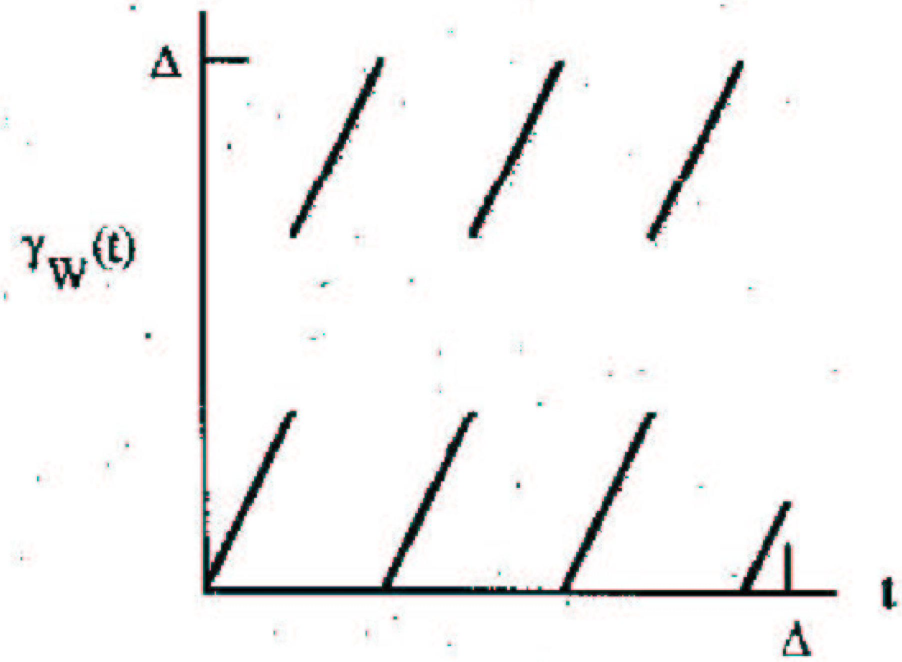

Fig. 4.

Appendix B. Paper #2 Figures

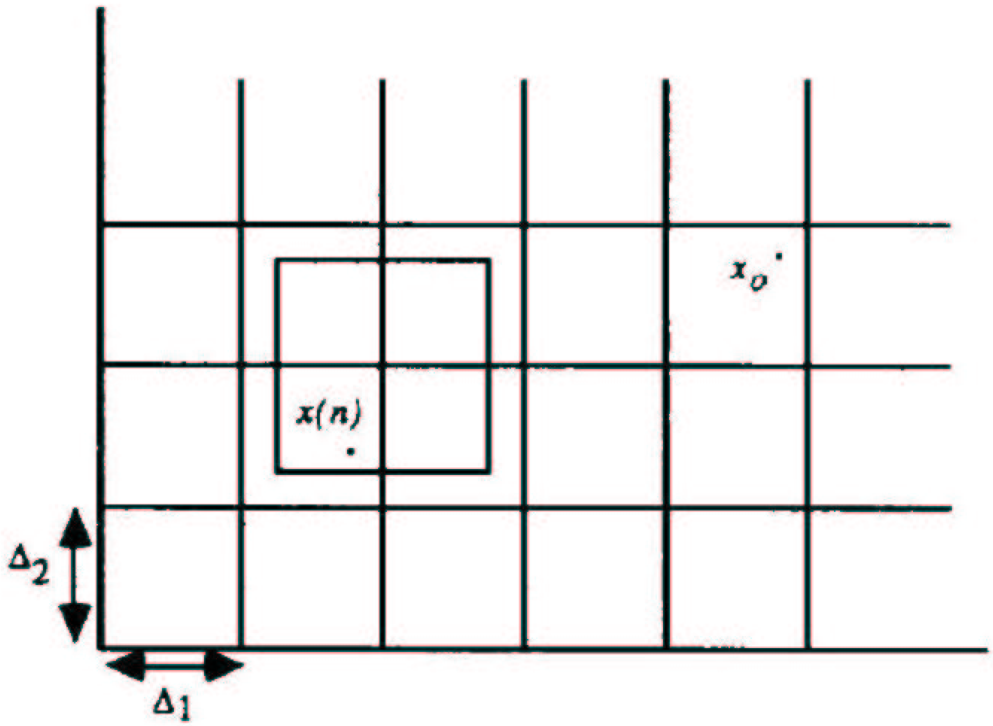

Fig. 1.

Figure 7: Visualization of state space partition for Proposition 2.2.

Paper #2 Figures (2)

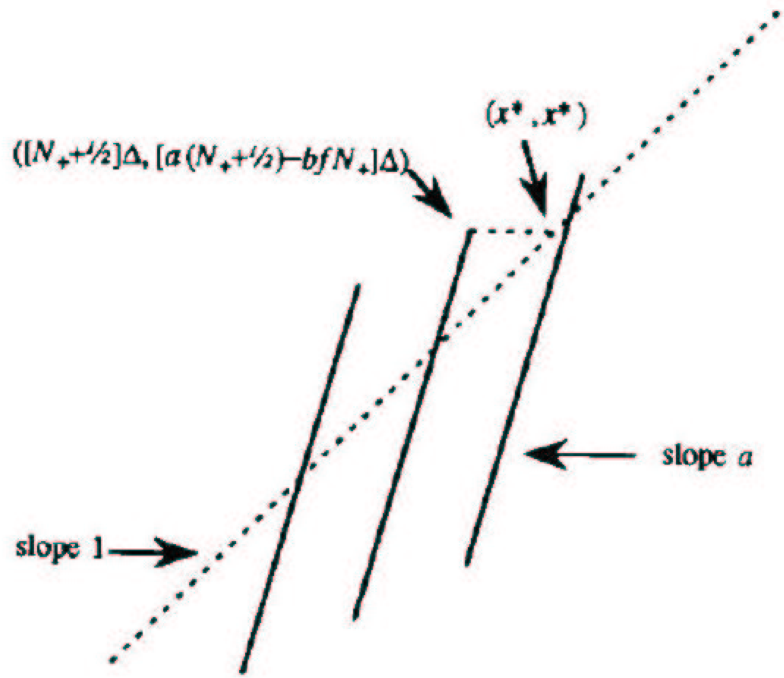

Fig. 2.

Figure 8: In support of proof of Lemma 3.1, (determining N_+ and x^*).

Paper #2 Figures (3)

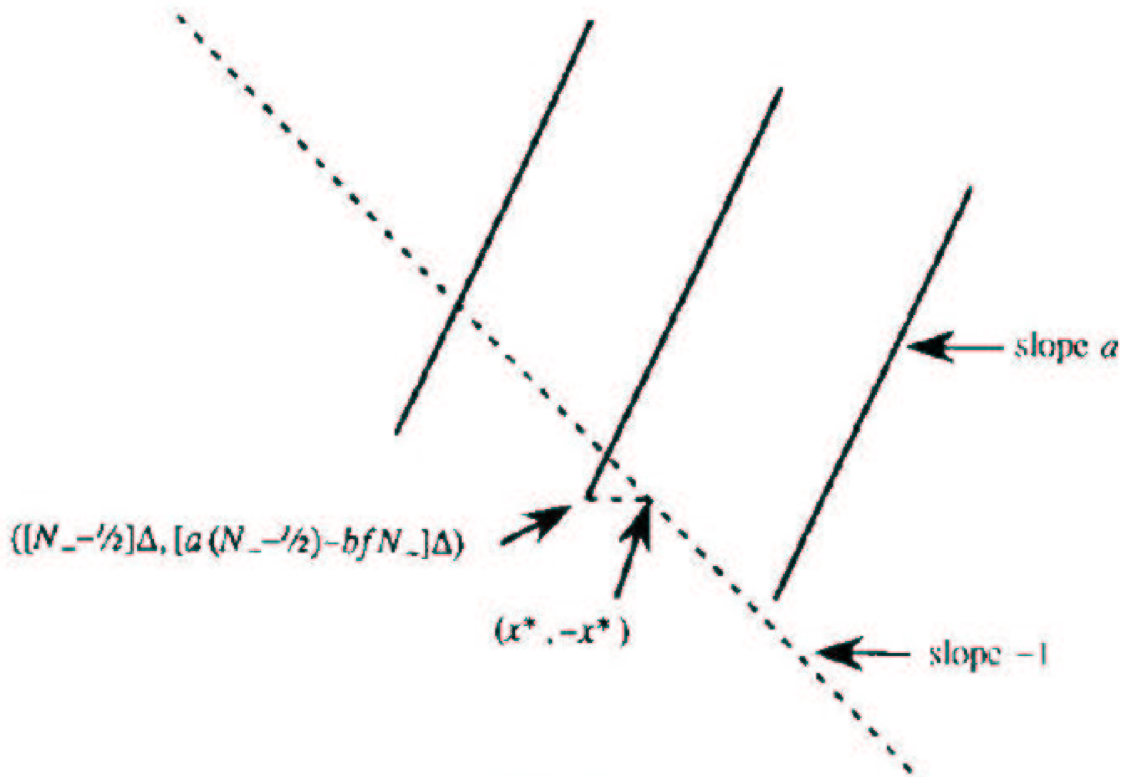

Fig. 3.

Figure 9: In support of proof of Lemma 3.1, (determining N_{-} and x^{*}).

Paper #2 Figures (4)

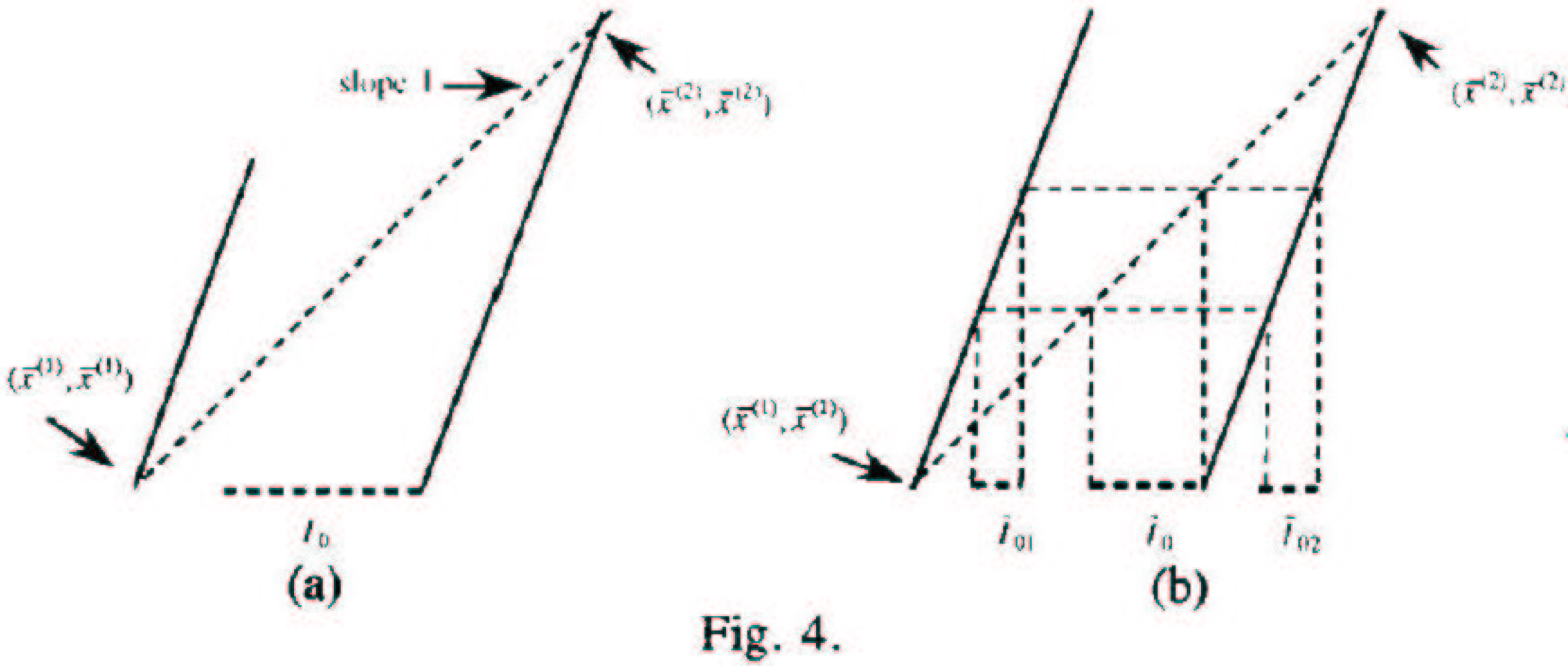

Figure 10: In support of Lemma 3.2

Paper #2 Figures (5)

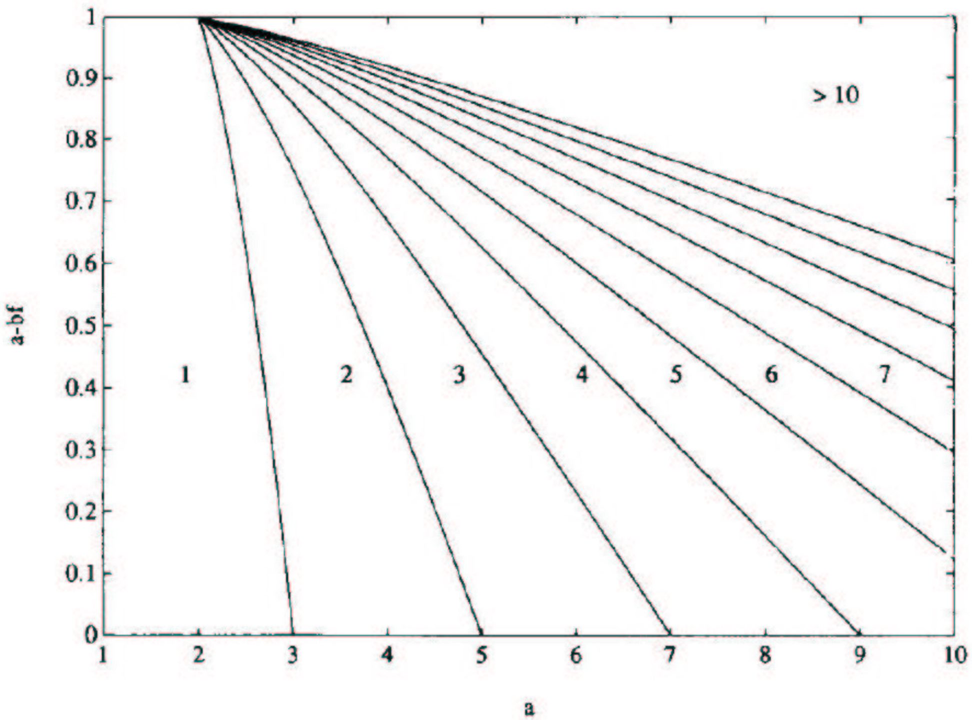

Fig. 5.

Figure 11: Parameter space for system (6). The numbers in the regions are the values of N_+.

Paper #2 Figures (6)

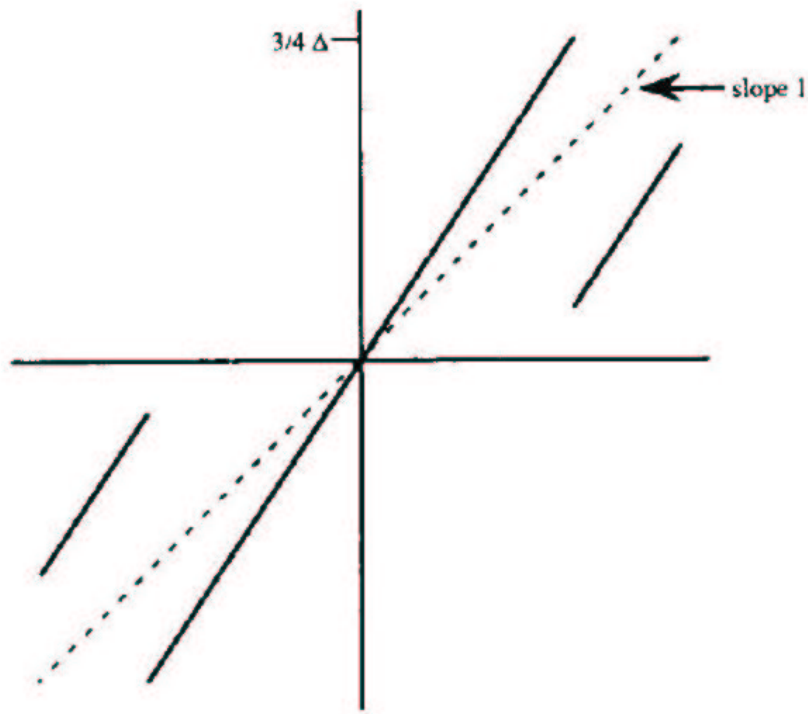

Fig. 6.

Figure 12: Example case of dynamics in (6) for $a = 3/2$, $bf = 5/8$, ($x^* = (3/4)\Delta$). Plot is of $G(x)$ versus x.