EE 150 Presentation

Week 5

David F. Delchamps: “Extracting state information from a quantized output record” (1989), and “Stabilizing a Linear System With Quantized State Feedback” (1990)

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Motivation

- We wish to control linear, time-invariant systems (P). Typically, \( x(\cdot) \in \mathbb{R}^n \), \( u(\cdot) \in \mathbb{R}^m \), \( y(\cdot) \in \mathbb{R}^p \). Unity feedback block diagrams shown, with controller C.

Figure 1: Continuous-time, \( t \in \mathbb{R} \).

Figure 2: Discrete-time, \( k \in \mathbb{Z} \).
• In quantized output systems, the output $y(\cdot)$ is restricted to a discrete, countable set of values, e.g. $y(k) \in Y \subset \mathbb{R}^p$ where $Y = \{y_i : i \in \mathbb{Z}\}$.

• In these papers, the output is taken to be a quantization of the state, i.e. the state passes through a quantizer $q(\cdot)$ to get to the output.

Figure 3: Discrete-time system with quantized output.
Motivation (3)

- Most modern control applications are implemented digitally.
- Quantization is an inherent part of digital implementation.

Figure 4: Typical controller implementation methodology.
Paper #1: Extracting state information


- **Underlying motivation**: Implementations of control technology becoming overwhelmingly digital rather than analog. Want to study interface between continuous systems and discrete implementation.

- Quantization often regarded as approximation of continuous measured state.

- Correlation of successive quantization errors usually not considered.

- **Idea**: Take quantized output *record* of a system to obtain a “better” determination of the system’s state.
Paper #1: Questions

- How much information about the current state of a given system is contained in a long record of past quantized measurements of the system’s output?

- How can we manipulate the system’s input to make the output record more informative about the state evolution?

Form of questions reminiscent of idea of *observability* in control theory.
Domain of consideration

- Discrete-time, linear, time-invariant (LTI) state-space systems

\[ x(k + 1) = ax(k) + bu(k), \quad k \geq 0 \]  \hspace{1cm} (1a)
\[ y(k) = q(x(k)) \]  \hspace{1cm} (1b)

- In general, \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p \)

- In this paper, consideration restricted to case where the state dimension is one \((n = 1)\) and the single-input, single-output (SISO) case \((m = 1, p = 1)\):
\[ x(\cdot) \in \mathbb{R}, \ a, b \in \mathbb{R}, \ u(\cdot), y(\cdot) \in \mathbb{R} \]
Quantization

- $q(\cdot)$ is a mapping from state space ($\mathbb{R}$) into a countable set $Y = \{y_i : i \in \mathbb{Z}\}$
- $q$ induces a partition on the state space, $\{U_i\}$, where $q(x) = y_i, \forall x \in U_i$.
- **Uniform quantizer with sensitivity $\Delta$:**

\[
q_\Delta(x) = \begin{cases} 
  i & \text{when } x \geq 0 \text{ and } x \in [(i - \frac{1}{2})\Delta, (i + \frac{1}{2})\Delta) \\
  -q_\Delta(-x) & \text{when } x < 0 
\end{cases}
\]
Figure 5: Uniform quantizer with sensitivity $\Delta = 0.4$. 
Output record

- Recall the output equation $y(k) = q(x(k))$
- Define the output record at time $k$ as $\{y(l) = q(x(l)) : 0 \leq l \leq k\}$.
- Define admissible control strategies as those that depend only on the current output record, i.e.
  $$u(k) = f^{(k)}(y(0), \ldots, y(k)), \quad k \geq 0$$
- (Note that this allows time-dependence for controller).
Symbolic dynamics

- Central idea: Assign a symbol to each element of the partition \( \{U_i\} \) (e.g. \( \{A, B, C, \ldots\} \)).

- Using a given fixed control strategy, evolution of state describes a symbol sequence associated with its output record.

- Symbolic dynamics seeks to relate the behavior of the closed-loop dynamics to the sequence of symbols that contain information about the trajectory of \( x(k) \).

- In particular...
Symbolic dynamics: questions

- Given a system of the form $x(k + 1) = G(x(k))$,

- To what extent does the symbol sequence associated with $x(0)$ determine $x(0)$?

- Under what conditions on the mapping $G$ and the chosen state-space partitioning can we make an “asymptotically perfect” determination of $x(0)$?
Example: Binary shift transformation

- Recall

\[ x(k + 1) = ax(k) + bu(k), \quad k \geq 0 \]
\[ y(k) = q(x(k)) \]

- Consider the case where \( a = 2, b = 1, q(x) = \text{floor}(2x), u = -y: \)

\[ x(k + 1) = 2x(k) + u(k), \quad k \geq 0 \]
\[ y(k) = \text{floor}(2x(k)) \]
Example: Binary shift transformation (2)

• $a = 2, b = 1, q(x) = \lfloor 2x \rfloor, u = -y$:

$$x(k + 1) = 2x(k) + u(k), \quad k \geq 0$$

$$y(k) = \lfloor 2x(k) \rfloor$$

• This map produces the binary expansion of $x(0) \in [0, 1]$, therefore $x(0)$ can be determined to arbitrary precision.

• Recall, binary expansion of $x$ is representation $\{a_i\}$ ($a_i \in \{0, 1\}$) such that

$$x = \sum_{i=1}^{\infty} \frac{a_i}{2^i}$$
Feedback regulation of information flow

- In the previous example, information about the *initial condition* is attained by looking at the output record.

- We want to use the same ideas to better estimate the *current* state.
Problem setup

- Assume $x(0)$ is uniformly distributed over the quantization interval $[(y_0 - \frac{1}{2})\Delta, (y_0 + \frac{1}{2})\Delta]$.
- Once the control law is chosen, $x(k)$ and $y(k)$ evolve as random processes.
- We wish to minimize our uncertainty about $x(k)$ given the output record $\{y(l) : 0 \leq l \leq k\}$.
Preliminaries and notation

- Let $y_0^k$ be shorthand for “$y(0) = y_0, \ldots, y(k) = y_k$”
- Let $f(x(k)|y_0^k)$ be the conditional probability density of $x(k)$ given $y_0^k$.
- Define the differential entropy, or dispersion of $x(k)$ as

$$h(x(k)|y_0^k) = -\sum_{(y_0, \ldots, y_k) \in Y^k} \text{prob}(y_0^k) \left( \int f(x(k)|y_0^k) \log f(x(k)|y_0^k) \, dx(k) \right)$$

- The lower this value, the more information about $x(k)$ is contained in the output record.
- $h(x(0)|y(0)) = \log \Delta$; and, generally, $h(x(k)|y(k)) = \log \Delta$. 
Given \( x(k + 1) = ax(k) + bu(k) \) \hspace{1cm} (1a) 
\[
y(k) = q(x(k)), \hspace{1cm} (1b)
\]

and a family of (admissible) control laws

\[
u(k) = f^{(k)}(y(0), \ldots, y(k)), \hspace{1cm} k \geq 0 \hspace{1cm} (2)
\]

**Theorem (3.1).** Assume in (1) that \( b \neq 0 \).

(a) If \( |a| < 2 \), then there exists a feedback law of the form (2) that makes

\[
h(x(k)|y_0^k) \to -\infty \text{ as } k \to \infty.
\]

(b) If \( |a| \geq 2 \), then a control law of the form (2) can be chosen so that \( h(x(k)|y_0^k) \) approaches a finite limit smaller than \( \log \Delta \).
Proof of central result, part (a)

- Idea: Use control to keep moving $x(k)$ to the junction of two quantization blocks.
- Shown that given this control, $h(x(k) | y_0^k) = k \log(\frac{1}{2}|a|) + \log \Delta$. 
Proof of central result, part (b)

- Relies on theory of Markov chains with countable state spaces.

- Idea: Given a control strategy, $x(k)$, given $\{y(l) : l \leq k\}$, is distributed uniformly over an interval of random length $\Lambda(k)$.

- For given control law, $\{\Lambda(k)\}$ is a Markov chain with countable state space $L$.

- A stationary distribution exists for this Markov chain, and that this limiting distribution results in convergence of $h(x(k)|y_0^k)$ to a value smaller than $\log \Delta$. 
Paper #2: Stabilizing a quantized system


- **Domain**: (Unstable) discrete-time, linear time-invariant (LTI) system with quantized measurements.

- **Problem statement**: Stabilize the system, that is, find a control law that brings closed-loop trajectories arbitrarily close to the origin for some time.

- Similar idea as paper #1: Consider quantized measurements as partial observations rather than approximations. Use a record of these partial observations to form better state estimates.
Paper #2: Questions

- Under what circumstances and in what sense can we stabilize an unstable LTI system using control based on only past quantized measurements?

- How does the answer to the above question depend on the properties of the system versus properties of the quantizer?
Quantization \( q \) maps to a countable set \( J \) (same as \( Y \) in paper #1)

System equation

\[
x(k + 1) = Ax(k) + Bu(k), \quad x(0) = x_0, \quad (x \in \mathbb{R}^n, u \in \mathbb{R}^m)
\]  

Admissible control

\[
u(k) = f^{(k)}(q(x_0), q(x(1)), \ldots, q(x(k)))
\]  

Let \( 0 \in \mathbb{R}^n \) lie in the interior of \( U_0 = q^{-1}(q(0)) \subset \mathbb{R}^n \), bounded.
Result: No asymptotic stabilization

**Proposition 2.1.** In (1), suppose that $A$ is unstable. Then for every control law of the form (2), the set of all $x_0 \in \mathbb{R}^n$ whose closed-loop trajectories $k \rightarrow x(k)$ tend to zero as $k \rightarrow \infty$ has Lebesgue measure zero.

• In what sense, then (not asymptotically), can we stabilize system (1)?

• From here in, consider only rectilinear uniform quantizers, i.e.
given $\Delta_1, \ldots, \Delta_n > 0$,

$$[q\Delta(x)]_j = q\Delta_j(x_j)$$

(Partitions $\mathbb{R}^n$ into rectilinear quantization “blocks” whose edges are parallel to co-ordinate axes).

• **Question:** Given $\epsilon > 0$, can we design a control law that get each closed-loop trajectory to get within $\epsilon$ of the origin and stay there an arbitrarily long time $K_o > 0$?
Result: Finite-time stabilization to an $\epsilon$-ball

- For system (1), let $q$ be the directionwise uniform quantizer $q_\Delta$ as defined. Let $\Delta_{\text{max}}$ and $\Delta_{\text{min}}$ denote the largest and smallest of the $\{\Delta_j\}$

Proposition 2.2. Suppose that $\|A\|_\infty \leq (2\Delta_{\text{min}}/\Delta_{\text{max}})^{1/n}$ and that $(A, B)$ is controllable. Then for every $\epsilon > 0$ and integer $K_0 > 0$ there exists a $K_1 > 0$ and an admissible control law such that all closed-loop trajectories are within $\epsilon$ of the origin for times $k \in [K_1, K_1 + K_0]$. 
• Design $F$ such that $(A - BF)$ is stable, and choose control law

$$u_i(k) = -F \bar{x}(k),$$

(3)

where $[\bar{x}(k)]_j = \Delta_j [q\Delta(x(k))]_j$.

• $\bar{x}(k)$ describes the center of the quantization block corresponding to $x(k)$.

**Proposition 2.3.** Let $\gamma$ be such that $\lambda_{\text{max}}(A - BF) < \gamma < 1$ and let $\Delta_{\text{max}}$ be the largest of the $\{\Delta_j\}$. Then there exists an ellipsoid $D$ centered at $0 \in \mathbb{R}^n$ and $\exists N(x_o) > 0$ for which $x(k) \in D$ for all $k \geq N$. If $x_o \in D$, then $N = 0$. 
Naive stabilization: analysis

- Dynamics (1) subject to control (3) yield closed-loop system:

\[ x(k + 1) = G(x(k)) = Ax(k) - BF\bar{x}(k) \]  

- Defining \( e(k) \triangleq x(k) - \bar{x}(k) \),

\[ x(k + 1) = (A - BF)x(k) + BFe(k) \]
Naive stabilization: special case

- Restrict consideration to state dimension $n = 1$:

$$x(k + 1) = ax(k) + bu(k), \quad k \geq 0$$

$$a \in \mathbb{R}, |a| > 1, b \in \mathbb{R}^{1 \times m}$$

- Let $f \in \mathbb{R}^m$ be such that $|a - bf| \leq 1$, and (given $\Delta$ and uniform quantizer $q_\Delta$) choose control

$$u(k) = -f \Delta q_\Delta(x(k)), \quad k \geq 0$$

$$\Rightarrow x(k + 1) = ax(k) - bf \bar{x}(k)$$

(6)
Naive stabilization in 1D: invariant sets

- Applying proof of Proposition 2.3, we find that the following $D$ is an invariant set of the closed-loop dynamics:

$$D = \left\{ x \in \mathbb{R} : |x| \leq \frac{|bf| \Delta}{2(1 - |a - bf|)} \right\}$$

- **Example**: $a = 3/2$, $bf = 5/8$, $(a - bf = 7/8)$, $D = [-2.5, 2.5] \Delta \subset \mathbb{R}$.

- This is not, however, the smallest invariant set.
Result: smallest symmetric invariant set

- Let $x^* \in \mathbb{R}$ denote $\inf_{x > 0} \{ x | G([-x, x]) \subset [-x, x] \}$.

**Lemma 3.1.** Suppose $a > 1$ and $a - bf \in [0, 1)$, then for the closed loop dynamics of (6),

\[
x^* = G(N_+ + \frac{1}{2}) = \left[ \frac{1}{2}a + N_+(a - bf) \right] \Delta,
\]

where

\[
N_+ = \min \left\{ N \in \mathbb{Z} : N \geq \frac{a - 3 + 2a^{-1}(a - bf)}{2[1 - (a - bf)]} \right\}.
\]

- **Example:** $a = 3/2$, $bf = 5/8$, $N_+ = 0 \geq -0.0208 \Rightarrow x^* = \frac{3}{4} \Delta$. 
Result: $D^*$ is almost globally attracting

\[ x(k + 1) = ax(k) - bf\bar{x}(k) \]  \hspace{1cm} (6)

Lemma 3.2:

- If no fixed points exist outside of $D^* = [-x^*, x^*]$, then every trajectory of (6) enters $D^*$ eventually.

- If fixed points exist outside of $D^*$, then these are unstable. The set of initial conditions whose trajectories never enter $D^*$ is contained in a set of measure zero.
**Result: invariant measures**

**Theorem 4.3.** Assume in (6), suppose that $a \in \mathbb{Z} : a \geq 2$ and that 

$$(a - bf)(N_+ + 1) \leq \frac{1}{2}a.$$ 

Then

i) there exists on $D^*$ exactly one probability measure $\mu^*$ that is both invariant under $G$ and absolutely cts. w.r.t. normalized Lebesgue measure $\lambda$ on $D$. The density of $\mu^*$ is positive $\lambda$-almost everywhere in $D^*$, and $G$ is ergodic w.r.t. $\mu^*$;

ii) almost every trajectory of (6) is dense in $D^*$; and

iii) $\lim_{L \to \infty} \frac{1}{L} \sum_{k=0}^{L-1} P^k_G(\phi) = \phi^*$

- $\phi^*$ is the density of $\mu^*$ w.r.t. $\lambda$

- $\phi \in L^1(\lambda)$ is the density of an arbitrary probability measure on $D^*$ absolutely continuous w.r.t. $\lambda$. 

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Central themes

- Quantization can be viewed as more than just an approximate measurement.
- Quantization can be taken into account explicitly during control system design, to achieve better closed-loop performance.
- Long-term behavior of quantized systems can be quite different from that expected by treating quantization as a noisy process.
Figure 6: Graph of binary shift transformation, $G(x) = \text{mod}(2x, 1)$, $x \in [0, 1]$. 
Paper #1 Figures (2)

Fig. 2.
Fig. 3.
Paper #1 Figures (4)

Fig. 4.
Figure 7: Visualization of state space partition for Proposition 2.2.
Figure 8: In support of proof of Lemma 3.1, (determining $N_+$ and $x^*$).
Figure 9: In support of proof of Lemma 3.1, (determining $N_-$ and $x^*$).
Figure 10: In support of Lemma 3.2
Figure 11: Parameter space for system (6). The numbers in the regions are the values of $N_+$. 
Figure 12: Example case of dynamics in (6) for $a = 3/2$, $bf = 5/8$, ($x^* = (3/4)\Delta$).

Plot is of $G(x)$ versus $x$. 