Separation of Estimation and Control for Discrete Time Systems

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Invited Paper

Abstract—An attempt is made to coordinate the numerous results relating to separation of estimation and control in discrete time stochastic control theory. The results vary widely depending upon the assumptions about linearity, criteria, information pattern, constraints, and noise distributions. Some of the less well-known underlying concepts are discussed with the help of a fairly general model.

I. Introduction

When James Watt invented the steam-engine governor he solved a problem that was due to unpredictable variations in the load and boiler conditions. Ever since, control engineering has had as its main objective the ability to cope with uncertainty. This is a problem to which statisticians and, later, decision theorists have given a great deal of attention.

The most important approach available is the one which, in spirit, is attributed to Bayes. It requires that an assessment, possibly subjective, be made at the outset as to the probabilities of the various uncertain quantities. Furthermore, a utility function must be set up to express in a consistent way the preference relations among the possible outcomes. Then a policy is optimal in a given set of possible policies if the expectation of the utility as computed with the given policy is as large as with any other possible policy.

In control engineering one talks of control laws instead of policies and one is to minimize expected cost rather than to maximize expected utility. These are only verbal changes, but a genuine complication results from the continuity of the time variable in dynamic systems described by (say) differential equations. The present paper is restricted to discrete time systems for two reasons. First, as soon as inputs or outputs or both are only updated at intervals (and this is common in practice) one is reduced in effect to a discrete time situation.

Second, the problem of stochastic control of continuous time systems is not as yet formulated in a precise and agreed upon fashion. Claims of optimality of a control law refer to a comparison with all other possible control laws. There is no agreement as to the definition of the latter set. One would wish to include all measurable and nonanticipative mappings of available data into controls. This leads to a system of nonlinear stochastic functional differential equations. An interpretation of these equations is required such that for every control law one obtains a solution leading to one and only one value of the expected cost. There is not as yet an agreement on how this can be done, though work is in progress [13], [116] toward that goal. In the meantime a claim of optimality for a continuous time stochastic control law is valid only under artificial restrictions, or else is more in the nature of a conjecture.

This paper is restricted to discrete time systems with a fixed finite number of stages, the so-called "finite horizon" case.[1] The infinite horizon case entails additional technical difficulties which are of limited importance to civilizations with limited life span, though it may be a useful approximation to a distant horizon.

Under these restrictions, the paper is an attempt to clear up some of the confusion surrounding the subject and to summarize some of the results that can be associated with the vague concept of a "separation" between estimation and control.

Because of the feedback nature of control and the importance of the information pattern it is paramount to distinguish sharply between two aspects of the word "control." The first aspect is the control *law* which maps available data into values of control inputs to the system. The control law is a function which is selected from a set of functions (infinite dimensional!), and there is nothing stochastic or uncertain about this function. The problem is to make an optimal selection. This is to be done by the designer before the system is put together.

The second aspect is the realization of the control *variables* actually applied to the system in operation. When the control laws have been selected and instrumented, and only then, the control variables (and the state and output variables, and the cost) *become* random variables, that is, become functionally related to the given random variables (noise, initial state) and therefore become functionally related to the underlying probability space. But to the designer who is still seeking for good control laws and has not made a selection yet, the realizations of control are not even random variables. They are just "random variables to be" of yet uncertain status.[2]

The most popular results in control theory are those for unconstrained control of linear systems with Gaussian noise, quadratic criteria, and the classical information pattern. Much confusion has been abetted by the incredible robustness of this case to conceptual misunderstandings: every reasonable assertion about that case is true and, within wide limits, no amount of confusion can give an incorrect result. In fact, the most confused derivations of the correct results are also among the shortest.

This singular situation ceases to prevail as soon as the assumptions are weakened. One may note in passing that a similar singularity exists in zero-sum linear-quadratic dynamic game theory.

There is no generally accepted meaning for the word "adaptive" in control engineering. In stochastic control, the control law generates the control value as a function of all the available data. Nothing can be more adaptive than that [86], [92]. If one goes to non-Bayesian decision procedures, including possible randomization of the control law selection, one still will finally pick the control as a function of the same data, only the opinion as to which selection procedure is better has changed.

This paper contains no mathematical theorems. Instead, the results of interest are stated as "assertions." To turn such assertions into theorems would involve in some cases continuity, separability,

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[1] This includes systems with a variable number of stages (stopping times) as long as a finite upper bound on this number is known.

[2] This confusion also arises in the study of communication channels with feedback, where it was only recently resolved [46].

growth, or moment conditions. This would tend to obscure the exposition of the main ideas. Furthermore, even if theorems were appropriately formulated, one would often search in vain in the literature for actual proofs. Supplying such proofs remains as an unfinished task for which some of the concepts presented in the following sections may be helpful.

II. Problem Description

While more general problems can be considered [103], the following description includes many important practical situations as special cases.

Consider a system operating for T time steps. Observations are made at each step from M observation posts and control inputs are applied at each step from K control stations. Here $T>0$, $K>0$, $M\geq 0$ are given integers.

The operation of this system can be described chronologically as follows, with the symbols x_t, u_t^k, y_t^m denoting vectors of various given finite dimensions.

Generation of random initial state x_0.
Observation of outputs $y_1^1, \cdots, y_1^M$ at the posts.
Application of inputs $u_1^1, \cdots, u_1^K$ by the stations.
Transition to state x_1.
$\cdots$
Transition to state x_{t-1}.
Observation of outputs $y_t^1, \cdots, y_t^M$ at the posts.
Application of inputs $u_t^1, \cdots, u_t^K$ by the stations.
Transition to state x_t.
$\cdots$
Transition to state x_T.

The uncertainties concerning operation of the system are modeled by the following set of $1+T(M+1)$ independent random vectors with given probability distributions, called the "primitive random variables":

$$x_0;\, v_t,\, w_t^m\ (t = 1, \cdots, T;\, m = 1, \cdots, M).$$

The variables are related by the state transition equations

$$x_t = f_t(x_{t-1}, v_t, u_t^1, \cdots, u_t^k, \cdots, u_t^K), \quad \text{for} \quad t = 1, \cdots, T \tag{1}$$

and the observation equations

$$y_t^m = g_t^m(x_{t-1}, w_t^m), \quad \text{for} \quad m = 1, \cdots, M \quad \text{and} \quad t = 1, \cdots, T. \tag{2}$$

The cost is given by the expression

$$\sum_{t=1}^{T} h_t(x_t, u_t^1, \cdots, u_t^K). \tag{3}$$

In general, the given functions f_t, g_t^m, and h_t are only assumed (Borel) measurable. Often one assumes $h_t\geq 0$ to insure an unambiguous albeit possibly infinite value for the expected cost, though other assumptions can make this restriction on h_t unnecessary.

The system will be called *linear* when all the functions f_t and g_t^m are jointly linear in all their arguments, while no restriction on h_t is implied.

The system will be called *semilinear* if (1) has the form

$$x_t = \Phi_t(v_t)x_{t-1} + \phi_t(v_t, u_t^1, \cdots, u_t^K) \tag{4}$$

and (3) has the form

$$\sum_{t=1}^{T} [H_t x_t + \psi_t(u_t^1, \cdots, u_t^K)] \tag{5}$$

while no restriction on the functions g_t^m is implied.

Finally, one must specify the possible ways in which the variables u_t^k can be generated, that is, the possible control laws. To do this one specifies the following:

1) the data available as arguments of the law;
2) restrictions on the values taken (the range);
3) restrictions on the functional form of the law.

The specification of the data available as arguments is the *information pattern* of the problem. Because of its great importance a special notation will be devised for this purpose.

Define the following sets of pairs of indices. For $t=1, \cdots, T$,

$$Y_t = \{(\tau, m)|\tau = 1, \cdots, t;\, m = 1, \cdots, M\}. \tag{6}$$

For $t = 2, \cdots, T+1$,

$$U_t = \{(\tau, k)|\tau = 1, \cdots, t-1;\, k = 1, \cdots, K\}. \tag{7}$$

Also,

$$U_1 = \varnothing \quad \text{(the empty set)}.$$

A *data basis* at time t is a pair (A, B) where A is a subset of Y_t and B is a subset of U_t. The pair (Y_t, U_t) is the maximal data basis at time t. The array of vectors designated by (A, B) is denoted y_A, u_B where y_A is composed of the y_τ^m, $(\tau, m)\in A$ and u_B is composed to the u_τ^k, $(\tau, k)\in B$.

An *information pattern* is the assignment, to each index pair (t, k) in U_{T+1}, of a data basis at time t, denoted $(Y_{t,k}, U_{t,k})$.

This is interpreted to mean that the control applied by station k at time t is based on the arguments y_τ^μ with (τ, μ) in $Y_{t,k}$ and u_θ^κ with (θ, κ) in $U_{t,k}$. In other words, these are the past observations and past controls that are available to station k at time t when it is to apply control u_t^k.

Then the control equations are

$$\begin{aligned} u_t^k &= \gamma_t^k(\cdots, y_\tau^\mu, \cdots; \cdots, u_\theta^\kappa, \cdots) \\ &= \gamma_t^k(y_{Y_{t,k}}, u_{U_{t,k}}) \end{aligned} \tag{8}$$

where, for each (t, k) in U_{T+1}, γ_t^k is the *control law* for that index pair and the arguments are the ones specified by the information pattern.

In general, the range of γ_t^k is assumed unrestricted and as a function γ_t^k need only be (Borel) measurable.

Two exceptions of this will be considered.

1) The case of control constraints, where the range of γ_t^k must be contained in a given compact set Ω_t^k for each pair (t, k). Such constraint sets may be variable as long as they depend only on the data specified by $(Y_{t,k}, U_{t,k})$. For instance, an output variable may be the remaining fuel or energy.

2) The case of "linear" control, where γ_t^k must be jointly affine (linear plus a constant) in all its arguments.

III. Topics in the Study of Information Patterns

A. Some Types of Information Patterns

Given an information pattern, control station k is said to have *perfect recall* when either $T=1$ or else one has $Y_{t,k}\subset Y_{t+1,k}$, $U_{t,k}\subset U_{t+1,k}$ for $t=1, \cdots, T-1$.

This means that any data that the station had at some time remains available to it at any later time. Note that a station which fails to record some of the control variables it has applied may yet have perfect recall. However, if it does, then it can always reconstitute the forgotten controls from what it remembers.

An information pattern is said to be *classical* if all stations receive the same information (pattern independent of subscript k) and have perfect recall. This implies that, for any design, the σ-fields determined in probability space by data available at successive stages are nested.

An important type of patterns are the *delayed sharing patterns*, characterized as follows. Let $M=K$ (hence $Y_t=U_{t+1}$) and $n>0$. Let $Y_{t,k}$ consist of all pairs (θ, μ) with $\theta \le t-n$, $1 \le \mu \le K$ and the pairs (θ, k) with $t-n<\theta\le t$. Let $U_{t,k}$ contain the same pairs except for those with $\theta=t$.

Such a system can be said to consist of K stations with perfect recall, each having an associated observation post. A station receives the outputs from its observation post immediately and it remembers the controls that it applies. The stations share their data with a delay of n steps.

Note that the delayed sharing pattern is not classical when $K>1$.

For $K=1$ (n irrelevant) one has as a special case the usual stochastic control problem [3] with a single station which has available at any time all past outputs and controls. This will be called the *strictly classical pattern*.

For $T=1$ (n irrelevant, $K>1$) one has the so-called *team theory* [74], with K stations having different data and each acting once.[3]

For $n=1$ ($T>1$, $K>1$) one has the *one-step-delay sharing pattern*, about which more later.

For $n=T>1$ one has the problem of K stations without any "lateral" sharing of information.

It can be shown that any problem with classical information pattern is equivalent to a problem with strictly classical pattern, though additional state variables may be required. As an example, suppose $K=M=1$, $U_{t,1}=U_t$, $Y_{t,1}=Y_{t-1}(Y_{1,1}=\phi)$, that is, the outputs are communicated from the observation post to the control station with a one-step delay. Then this delay can be represented by means of additional state variables which propagate the variable y_{t-1} of the given system to appear as output at time t in the new system (take g_1^1 constant in the new system).

B. Field Bases and Conditioning

Let (Y, U) be a data basis at time t. Does it make sense to speak of the conditional distribution (or expectation) of x_{t-1} given (the data specified by) data basis (Y, U)? The answer is no, outside of certain special cases discussed later. The reason for this negative answer is that the control laws have not been specified. Therefore, neither x_{t-1} nor the variables designated by (Y, U) are random variables. Information about the control laws used is required, the question is only: how much information? To discuss this, appropriate notation must be introduced.

The designer must ultimately choose a functional form $\gamma_t^k(\cdot)$ measurable on the data specified by $(Y_{t,k}, U_{t,k})$ for each pair (t, k) in U_{T+1}. A choice of a control law γ_t^k for each pair (t, k) in U_{T+1} will be called a design. Denote by Γ the set of all possible designs. Then an element γ in Γ provides maximal information about the control laws. If γ is given, all the system variables become random variables and, in particular, a data basis (Y, U) will determine a σ-field in probability space, which may be denoted $\mathcal{F}(Y, U; \gamma)$. Now a regular conditional distribution of x_{t-1} (and of any other variable in the system) given this field is well defined, by virtue of Theorem 27.2.B of Loève [111, p. 363]. What we must investigate is the possibility that it remains well defined when information about some of the control laws is withheld.

For L a subset of U_{T+1} define γ_L to be the specification of the functional form of just those control laws $\gamma_\theta^k(\cdot)$ with subscripts (θ, k) in L.

A triple (Y, U, L) with Y a subset of Y_t and U, L subsets of U_t is called a *field basis* (at time t) if for any two designs γ, $\bar{\gamma}$ in Γ the relation $\gamma_L=\bar{\gamma}_L$, that is, $\gamma_\theta^k(\cdot)\equiv\bar{\gamma}_\theta^k(\cdot)$ for (θ, k) in L, implies that the fields $\mathcal{F}(Y, U; \gamma)$ and $\mathcal{F}(Y, U; \bar{\gamma})$ are the same.

In other words, knowledge of just those control laws designated by L is sufficient to determine unambiguously the field generated by the variables designated by (Y, U). Now consider a system variable z and note that the choice of design influences in general both the field determined by (Y, U) and the relation of z to the probability space. Therefore, a field basis does not necessarily determine a conditional distribution. On the other hand, a basis which is not a field basis can be sufficient to determine a conditional distribution. This motivates the next definition, where conditional distributions of a random variable z will be considered as functions from the probability space into the space of probability distributions for z.

The triple (Y, U, L) is a *conditioning basis* for a variable z if there exists a function F, such that for any design γ the conditional distribution of z given data (Y, U), that is, given the field $\mathcal{F}(Y, U; \gamma)$, is almost surely $F(y_Y, u_U, \gamma_L)$.

Observe that if ω denotes the array consisting of all primitive random variables, then a field basis is automatically a conditioning basis for ω, but not conversely. For any subset L of U_{T+1} the triple $(\varnothing, \varnothing, L)$ is a field basis, regardless of the information pattern of the problem. Also, for any $Y\subset Y_t$, $U\subset U_t$ and any information pattern, the triple (Y, U, U_t) is a field basis and a conditioning basis for x_{t-1}, by causality. If, in particular, $Y=Y_{t,k}$, $U=U_{t,k}$ this just means that station k can determine at time t the conditional distribution of the latest state x_{t-1}, given the data it has available, provided it is allowed to use in this calculation information about any control laws implemented by any station at earlier stages.

C. Equivalences and Substitutions

In the design and analysis of stochastic control systems, substitutions between various types of data are often made. They are based upon equivalences between different formulations. Some of these equivalences are discussed here.

First, observe that any system variable z can be considered as a given function of ω and u, where ω is the array of all primitive random variables and u the array of all control variables. That is, $z=f(u, \omega)$, where f is obtainable by using (1) and (2) and is independent of the design γ and of the information pattern. For any design γ the dependence upon u can be eliminated by recursive substitution in the system equations. This solution process [103] turns the control variables into random variables by a substitution

$$u = S(\omega; \gamma). \tag{$*$}$$

Using this solution process, every system variable $z=f(\omega, u)$ becomes a parametric family of random variables

$$z = f(\omega, S(\omega; \gamma)) \equiv g(\omega; \gamma)$$

parametrized by the design γ.

Note that the solution process $(*)$ is defined for all designs γ in which each γ_t^k is measurable on (Y_t, U_t) the maximal data basis at time t. *A fortiori*, it is defined for all designs using any particular causal information pattern.

Two *designs* γ, γ^* are called *equivalent* when $S(\omega; \gamma)=S(\omega; \gamma^*)$ for almost all ω. This implies that for any system variable, such as z above, one has almost surely $g(\omega; \gamma)=g(\omega; \gamma^*)$. Thus the joint distribution of all system variables is the same with either design.

Two *information patterns* $(Y_{t,k}, U_{t,k})$, $(Y_{t,k}^*, U_{t,k}^*)$ are called *equivalent* when for any design feasible with the first pattern there is an equivalent design feasible with the second pattern and vice versa.

[3] A more general, informal notion of team theory had been proposed earlier [58].

In general, the data available for control do not form a field basis. However, for linear systems with strictly classical pattern, one has an exceptional situation illustrating the following assertion.

Assertion 1: If, for every (t, k), $(Y_{t,k}, U_{t,k}, \phi)$ is a field basis, then the given feedback control problem is equivalent to a feedforward control problem.

A feedforward control problem is one in which the data available depend only on the primitive random variables ω and not upon the control variables applied [13]. Such an equivalence plays a key role in some of the separation results for classical linear systems [78].

A more common type of equivalence is the following.

Assertion 2: Suppose that for some pair (t, k) there is a function ϕ such that, for all ω and γ,

$$(y_{Y_{t,k}}, u_{U_{t,k}}) = \phi(y_Y, u_U, \gamma_{U_t})$$

with $Y \subset Y_{t,k}$, $U \subset U_{t,k}$. Then the given pattern is equivalent to the one in which $(Y_{t,k}, U_{t,k})$ is replaced by (Y, U).

This can be seen from the substitution

$$\gamma_t^{*k}(y_Y, u_U) = \gamma_t^k(\phi(y_Y, u_U, \gamma_{U_t}))$$
$$\gamma_\tau^{*\kappa} = \gamma_\tau^\kappa, \quad \text{for } (\tau, \kappa) \neq (t, k)$$

noting that

$$\gamma_{U_t}^* = \gamma_{U_t}.$$

In particular, a station with perfect recall need not store the values of the control variables that it generates. However, both the form of an optimal policy and its determination may be simpler when explicit dependence upon past controls is allowed. Essentially, this is so because dependence upon the values of control variables can make relevant conditional distributions independent of the corresponding control laws [19].

Two *conditioning bases* (Y, U, L), (Y^*, U^*, L^*) for a variable z are called *equivalent* if for all designs γ feasible with the information pattern, and for almost all ω, one has agreement of the conditional distributions. That is,

$$F(y_Y, u_U, \gamma_L) = F^*(y_{Y^*}, u_{U^*}, \gamma_{L^*})$$

where both sides are distribution valued functions of ω and γ.

To decrease the reliance upon knowledge of previous control laws one might at first be tempted to invoke the following *incorrect substitution principle:* if (Y, U, L) is a conditioning basis for z and (t, k) belongs to both U and L, then $(Y, U, L-\{(t, k)\})$ is an equivalent conditioning basis. In fact, one must take into account the arguments of γ_t^k as specified by the information pattern. If they are not among the available data, then simultaneous knowledge of the value u_t^k and the law γ_t^k may provide valuable inferences about these arguments which would be impossible if either the value or the law were unknown. The correct substitution principle is as follows.[4]

Assertion 3: Suppose (Y, U, L) is a conditioning basis for z and one has $(t, k) \in U \cap L$, $Y_{t,k} \subset Y$, $U_{t,k} \subset U$. Then (Y, U, L), $(Y, U-\{(t, k)\}, L)$, and $(Y, U, L-\{(t, k)\})$ are equivalent conditioning bases for z.

Using the substitution principle, one can sometimes obtain conditional distributions that are independent of the design. The most important situations of this kind are special cases of the following assertion, where $L_{t,k}^n = \{(\theta, \kappa) \in U_t | \kappa \neq k,\ t-n < \theta < t\}$. (For $K=1$ or $n=1$, $L_{t,k}^n = \varnothing$.)

Assertion 4: For an n-step delayed sharing pattern and any $(t, k) \in U_{T+1}$ the triple $(Y_{t,k}, U_{t,k}, L_{t,k}^n)$ and the triple $(\bigcap_{k=1}^K Y_{t,k}, \bigcap_{k=1}^K U_{t,k}, \varnothing)$ are both conditioning bases for x_{t-n}.

[4] See [19] for an early appearance of the idea involved here.

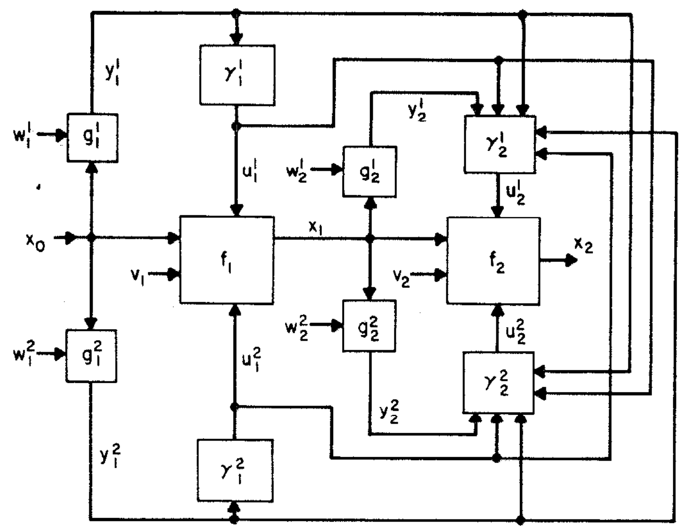

Fig. 1.

Note that the two bases mentioned in Assertion 4 are not equivalent when $K>1$, $n>1$.

As a special case, for $n=1$ the distribution of the latest state x_{t-1} given the data available to station k at time t is independent of the design. This fact is the keystone of much of the existing stochastic control theory. The case $n=1$ includes the strictly classical pattern [87] and (trivially) team theory.

D. Data Flow Diagrams

The diagram in which the plant and the controller are each represented by a "black box" does not convey anything about the prevailing information pattern. The more detailed diagrams required to do this become rapidly unwieldy but there is a certain didactic value in drawing them for simple cases. They are explicit data flow diagrams. In these diagrams a box represents a function and lines carry values of functions that may appear as arguments (inputs) to other functions (boxes). The essential point is that a box may not be used more than once, that is, each time step has a separate set of boxes. Thus in general there will be T boxes for (1), TM boxes for (2) and TK boxes for (8). The latter set of boxes is to be "filled" with admissible functions γ_t^k by the designer. The input lines to these boxes represent graphically the information pattern $(Y_{t,k}, U_{t,k})$.

For example, consider a delayed sharing pattern with $n=1$, $K=T=2$ which leads to the diagram of Fig. 1. The primitive random variables appear as inputs. The control variables u_t^k do not appear as inputs but the control laws γ_t^k (which may be considered similar to programs to be loaded in the control computers) are inputs, though of a quite different kind, since they are put in by the designer before the system starts operating.

When specific systems are under discussion the data flow diagram may show, instead of a simple box for a function f, some details of the structure of function f using boxes for more elementary functions from which f is built up.

E. Alternative Formulations

An apparently more general formulation is obtained by taking (2) as

$$y_t^m = g_t^m(x_{t-2}, w_t^m, u_{t-1}^1, \cdots, u_{t-1}^K) \tag{2'}$$

for $t>1$.

Note that if one had x_{t-1} instead of x_{t-2} as argument here, substitution of (1) would immediately yield the form (2′).

An apparently more special formulation is the one where (2) is just

$$y_t^m = G_t^m x_{t-1} \quad (2'')$$

and G_t^m is the zero-one matrix which extracts a subvector consisting of some of the components of x_{t-1}.

These two formulations and the one given earlier are in fact equivalent because (2′) can be reduced to (2″) by the obvious device of including the vectors w_t^m in an expanded vector v_{t-1} and including the vectors y_t^m in an expanded vector x_{t-1} from which they are extracted as in (2″) while the relations (2′) become part of the expanded form of (1). The intermediate formulation that was chosen for (2) is only didactically motivated.

The form (3) for the criterion assumes that the cost is a sum of terms each dependent on variables at a single time stage. More general criteria can be considered, with terms whose arguments straddle several time stages. Such criteria can be reduced to form (3) by propagating the arguments for each term, up to the last stage involved, as part of an expanded state vector.

IV. Some Results Related to Separation

A. The General Separation for the Strictly Classical Pattern

With the strictly classical pattern the superscripts k or m can be suppressed. By Assertion 4 there exists a regular conditional distribution of x_{t-1} given the variables $(y_1, \cdots, y_t, u_1, \cdots, u_{t-1})$ independent of the design γ.

This conditional distribution can be interpreted as a function F_t such that $F_t(y_1, \cdots, y_t, u_1, \cdots, u_{t-1})$ is a probability measure on the space of x_{t-1}.

The evaluation of F_t is called the filtering problem and is usually accomplished recursively [87], [3], [17]. The important fact is that this filtering problem is entirely independent of the criterion and of the control laws.

Now the separation result can be stated as follows [87].

Assertion 5: In the above problem there is no loss[5] if one restricts the designs to be of the form $\gamma_t = \phi_t \ F_t$ for all t.

This means that one may seek, instead of γ_t, a function ϕ_t defined on the (infinite dimensional) space of probability distributions for x_{t-1}. Once ϕ_t has been found, one has

$$\gamma_t(y_1, \cdots, y_t, u_1, \cdots, u_{t-1}) = \phi_t(F_t(y_1, \cdots, y_t, u_1, \cdots, u_{t-1})). \quad (9)$$

This is a worthwhile result because the functions ϕ_t can be determined as for a Markovian decision problem, by backward induction, with distributions playing the part of the state [87], [26]. Moreover, ϕ_t need only be defined on the range of F_t and there are cases where this range has finite dimension.

Once the γ_t have been found for all t, their dependence on the control variables u_t could be eliminated, if desired, by setting

$$\begin{aligned}\bar{\gamma}_1(y_1) &= \gamma_1(y_1)\\ \bar{\gamma}_2(y_1, y_2) &= \gamma_2(y_1, y_2, \bar{\gamma}_1(y_1))\\ \bar{\gamma}_3(y_1, y_2, y_3) &= \gamma_3(y_1, y_2, y_3, \bar{\gamma}_1(y_1), \bar{\gamma}_2(y_1, y_2)), \cdots, \text{etc.}\end{aligned} \quad (10)$$

Note, however, that the functions $\bar{\gamma}_t$ could not be as easily determined by backward induction because the conditional distribution of x_{t-1} given only $y_1, \cdots, y_t$ is not independent of the design of the earlier stages.

[5] A restriction from the set Γ of possible designs to one of its subsets Γ_r is said to be "without loss" when the infimum of the expected cost is the same whether taken over Γ or over Γ_r.

Note that no relation is being claimed between the functions ϕ_t and the feedback form of the optimal control of the deterministic system which results when the primitive random variables are fixed at their mean values. Note also that the "separation" is one-way: the filtering algorithm F_t must be determined first because the Markovian problem of finding the optimal ϕ_t has dynamics depending upon F_t.

B. Linear Gaussian Systems: Filtering

A random vector will be called Gaussian if its distribution is Gaussian, including the case of a singular covariance matrix, that is, of a distribution localized on a flat of lower dimension.

When a number of vectors are jointly Gaussian the (regular) conditional distribution of one of them given some of the other "observed" vectors has a Gaussian version with a covariance independent of the observed vectors and a mean which is jointly affine (that is, linear plus a constant) in the observed vectors [67].

This fact has immediate application to the filtering problem for linear Gaussian systems.

Assertion 6: With an n-step delayed sharing pattern, for any (t, k) in U_{T+1} the conditional distribution of x_{t-n} given the data designated by $(\bigcap_{k=1}^K Y_{t,k}, \bigcap_{k=1}^K U_{t,k})$ has a Gaussian version with covariance independent of the data and mean affine in the data.

Note that under these conditions the conditional covariance and the coefficients in the relation giving the conditional mean are independent of both the design (Assertion 4) and the data (Assertion 6). Whether or not the design γ is affine is irrelevant to the above results.

In general the conditional distribution of x_{t-n} given $(Y_{t,k}, \varnothing)$ or $(\bigcap_{k=1}^K Y_{t,k}, \varnothing)$ and the unconditional distribution are non-Gaussian and depend upon the design. If it is known that the design γ is affine, then these distributions are Gaussian but the mean and covariance depend in a complicated way upon the coefficients in the affine functions γ_θ^κ.

When γ is affine in the variables stipulated by the one-step-delay pattern (in particular, the strictly classical pattern), then recursive substitution (10) yields an equivalent design $\bar{\gamma}$ with dependence solely upon the *output* variables specified by the pattern. This design $\bar{\gamma}$ will still be affine.

Recursive organization of the calculation of the covariances and the coefficients in the expression of the mean is the subject of linear discrete time filtering [47]. The important feature about these calculations is that they can be done not only before the system is started but even before the criterion has been given and before a design has been selected. The calculation is finite dimensional. In the classical case it defines completely the functions F_t of Assertion 4. These functions have finite dimensional range which can be indexed by the mean. If $\bar{F}_t$ is the function giving this mean, then one has

$$F_t(y_1, \cdots, y_t, u_1, \cdots, u_{t-1}) = G(C_t, \bar{F}_t(y_1, \cdots, y_t, u_1, \cdots, u_{t-1})) \quad (11)$$

where $G(C, m)$ denotes the Gaussian distribution with mean m and covariance C, and C_t is the precomputed conditional covariance of x_{t-1}.

C. Linear Control of Linear Gaussian Systems with Arbitrary Pattern

Suppose that a linear system with Gaussian noise but general criterion is given and that only affine control laws are allowed. Then, for any information pattern, the problem can be reduced to one of optimal control of an entirely deterministic and still finite dimensional system, albeit a nonlinear one.

To accomplish this reduction one need only observe that for given affine control laws the states are affine functions of the primitive random variables. Hence the unconditional distribution of x_t is Gaussian and fully characterized by its mean and covariance. The components of the mean and the independent components of the covariance matrix can be considered as forming a vector and this is the state vector of the new system. The coefficients in the control laws are the control variables of the new system. The criterion becomes a deterministic function of these new state and control variables. The state equations for the new system are easily established from the properties of Gaussian distributions [62].

When the information pattern is nonclassical the nonlinearity of the new system makes it quite difficult to optimize. With quadratic criteria and the strictly classical pattern the cost is a convex function of the new control variables (the control law coefficients) and the best affine control law for the original system is then actually optimal among all measurable control laws. These two results both fail for nonclassical patterns. Even with quadratic criteria there may be several local minima of the cost as a function of the coefficients and the best affine control law may yield a cost several times as large as can be obtained by a good nonlinear control law [101].

D. Linear Gaussian Systems with Classical Pattern

Consider the control of a linear Gaussian system with the strictly classical pattern. No special assumptions are made about the criterion. The control variables may or may not be restricted to constraint sets. The functional form of a control law is unrestricted.

For this problem both the results of Section IV-A and those of Section IV-B apply, that is $\gamma_t = \phi_t \circ \bar{F}_t$ where $\bar{F}_t$ is the conditional mean of x_{t-1} given the data and ϕ_t can be determined by backward induction.

Thus $\bar{F}_t$ characterizes the filter part of the design and is affine, while ϕ_t is in general nonaffine and thus γ_t will be nonaffine. Note that in general ϕ_t does *not* coincide with the feedback form of the optimal control for the deterministic problem in which the random variables are held at their mean values.

The continuous time version of this separation theorem has been established under certain assumptions [106].

E. Linear Quadratic Systems with Classical Pattern

Consider a linear system with strictly classical pattern, quadratic criteria, and unconstrained control. The distributions of the primitive random variables need not be Gaussian.

In that case the conditional distributions $F_t(\cdots)$ of x_{t-1} are not generally Gaussian and their means $\bar{F}_t(\cdots)$ are not generally affine in the data.

Now consider the deterministic system obtained by fixing the primitive random variables at their mean values and let $\phi_t^*(x_{t-1})$ be the feedback form of the optimal control for this system at time t. Of course, ϕ_t^* is affine [7].

Assertion 7: In the above case the design

$$\gamma_t(y_1, \cdots, y_t, u_1, \cdots, u_{t-1}) = \phi_t^*(\bar{F}_t(y_1, \cdots, y_t, u_1, \cdots, u_{t-1})) \quad (12)$$

is optimal [77], [78]. In general γ_t will not be affine.

Note that since ϕ_t^* is affine it commutes with conditional expectations and one has $\gamma_t(\text{data}) = E\{\phi_t^*(x_{t-1}) | \text{data}\}$ a.s.

Thus one can say that the stochastically optimal control is the (conditional) mean of the deterministically optimal control. This situation is sometimes called "certainty equivalence" [91], [24].

F. Classical Linear Quadratic Gaussian Systems

The unconstrained control of a linear Gaussian system with strictly classical pattern and quadratic criterion is the most widely known case. The results of Sections IV-A, IV-B, IV-D, and IV-E all apply. They can be summarized as follows. The optimal control law γ_t is obtained by composition $\gamma_t = \phi_t^* \circ \bar{F}_t$ where both ϕ_t^* and $\bar{F}_t$ are affine. Hence γ_t is affine. Furthermore, the determination of ϕ_t^* and that of $\bar{F}_t$ can be accomplished entirely independently of each other. The filtering problem ($\bar{F}_t$) and the deterministic control problem (ϕ_t^*) have dual structure. Both depend on the solution of a similar type of matrix difference equations which, by analogy with the continuous time limit, are referred to as matrix Riccati equations.

This situation is essentially a coincidence since it depends not just on the linearity of the system but on the *Gaussian* nature of the noise and the *quadratic* nature of the criterion. If either of these is withdrawn, as in Sections IV-D and IV-E, the filtering and Markovian control problems become dissimilar.

G. Separation for Delayed Sharing Patterns

Consider the general system of Section II with the n-step delayed sharing pattern defined in Section III-A. Constraints on the control variables are allowed.

Denote by δ_t the data known to all K stations at time t, that is, the array $(y_i^k, u_i^k; k=1, \cdots, K; i=1, \cdots, t-n)$. The array is vacuous for $t \le n$. Denote by λ_t^k the additional data known to station k at time t, that is, the array $(y_i^k, u_j^k; i=t-n+1, \cdots, t; j=t-n+1, \cdots, t-1)$.

By virtue of Assertion 4 the conditional distribution of x_{t-n} given δ_t is independent of the design. Let $F_t(\delta_t)$ be this distribution, where F_t describes a filtering algorithm independent of the design. Then a separation type result can be stated as follows.

Assertion 8: In the above problem there is no loss if one restricts the designs to be of the form

$$\gamma_t^k(\delta_t, \lambda_t^k) = \phi_t^k(\lambda_t^k, F_t(\delta_t)). \quad (13)$$

The advantage is that F_t is an algorithm similar to the one used in the classical case and is independent of the criterion and the controller design, while the ϕ_t^k are determined by a backward induction where only n time stages, instead of T, are involved. In general, though, these "reduced" problems with n stages and K stations are still quite difficult. In the absence of sharing, one has $n=T$ and Assertion 8 is vacuous.

For linear systems with Gaussian noise the filtering problem is a linear one, the conditional distribution $F_t(\delta_t)$ is Gaussian with covariance independent of the data and mean $\bar{F}_t(\delta_t)$ affine in δ_t.

Assertion 9: For linear Gaussian delayed sharing patterns there is no loss if one restricts the designs to be of the form

$$\gamma_t^k(\delta_t, \lambda_t^k) = \phi_t^k(\lambda_t^k, \bar{F}_t(\delta_t)). \quad (14)$$

Thus the infinite dimensional space of distributions is avoided and the determination of the functions ϕ_t^k in (14) is accordingly simplified though the need to consider cooperation of K stations over n stages remains.

H. The Linear Quadratic Gaussian One-Step-Delay Problem

For a one-step-delay pattern one has $\lambda_t^k \equiv y_t^k$ and $F_t(\delta_t)$ is the distribution of x_{t-1} given the common data.

When the system is linear, the noise Gaussian (so that Assertion 9 applies), the criteria are quadratic and there are no constraints, then the following occurs.

Assertion 10: For the above problem there is an optimal design of the form (14) in which the functions ϕ_t^k, hence also the functions γ_t^k, are affine.

This result had been obtained earlier [74] for the case $T=1$, that is, for team theory. It extends by induction to the one-step-delay sharing case.

In a problem of this type one may, without loss, postulate an affine control law and the determination of the coefficients can be made by rational computations.

In contradistinction, for $n>1$ the assumption of an affine form for ϕ_t^k is not always optimal and, if it is made, the determination of the coefficients will usually require finding roots of algebraic equations of high degree [101].

I. The Semilinear Case

Consider a semilinear system described by (4), (2), and (5) with any given, not necessarily classical, information pattern. The control variables are restricted to given bounded sets Ω_t^k.

Let

$$\bar{\Phi}_t = E\{\Phi_t(v_t)\}$$
$$\bar{x}_0 = E\{x_0\}$$

and

$$\bar{\Phi}_t(u_t^1, \cdots, u_t^K) = E\{\phi_t(v_t, u_t^1, \cdots, u_t^K)\} \tag{15}$$

where the u_t^k are considered as constant in the expectation.

Consider the deterministic system with initial state $\bar{x}_0$, transition equation

$$x_t = \bar{\Phi}_t x_{t-1} + \bar{\phi}_t(u_t^1, \cdots, u_t^K) \tag{16}$$

and cost (5).

Assume that $\bar{u}_t^k$, $t=1, \cdots, T$; $k=1, \cdots, K$ is an optimal open-loop control schedule for this deterministic problem.

Assertion 11: The control laws $\gamma_t^k(\cdot) \equiv \bar{u}_t^k$ are optimal for the stochastic problem [66].

Thus in this special case data are useless. A similar situation prevails for linear minimax problems with linear criterion [104].

J. Separate Optimization of Measurement Systems

It may be possible in some situations to select at each step new values for some coefficients occurring in the output equations. In general, such a case can be reduced to the usual one by considering the output equations as part of the state equations, while the new output equations have the form (2″) of Section III-E. The coefficients in question then become simply control variables for which an information pattern must be given. Constraints on the coefficients are treated like any control variable constraints and costs associated with the selection of particular coefficients are added to the cost of control. The known results are then applied to the system that results.

In the special case of strictly classical unconstrained linear, Gaussian, quadratic systems (Section IV-F), consider the possibility of selecting coefficients of the affine output equations. (Adjustment of the output noise covariance can be modeled as a coefficient adjustment.) Constraints on the coefficients can be given as well as a cost (not necessarily quadratic) dependent on their values. The data available for selecting coefficients of g_t consist of all previous outputs, controls, and coefficients.

Clearly, if the coefficients are selected beforehand, then the control problem is that of Section IV-F which is solved on the basis of these coefficients and yields an expected cost of control which depends upon the coefficients. This dependence can be described in terms of the "Riccati" equation of the filtering problem. The *a priori* optimization of the coefficients is thus a deterministic problem with a nonlinear (Riccati) state equation. Such problems are by no means trivial, but have been solved. They have also appeared, in a similar fashion, when optimization of signals for communications purposes is considered [8].

In the case under discussion it has been shown [65] that this *a priori* optimization of the coefficients is optimal, so that the on-line data are useless as far as the selection of coefficients is concerned. In this respect the situation is similar to the one encountered in Section IV-I for semilinear systems.

If, in particular, observations are multiplied by a coefficient of 0 or 1, with cost for the choice of 1, one has the special case of optional costly observations. The result then states that the observation schedule can be fixed in advance without loss [4], [118].

K. Approximations to Optimality

Because of the complexities of optimal stochastic control of nonlinear systems, approximations are often the only recourse.

One of the more straightforward methods of approximation rests on the idea that any sufficiently smooth nonlinear system is linear (and the criterion quadratic) within higher order terms, in the neighborhood of a reference sequence of states. A control scheme is then designed using the separation properties of linear quadratic systems.

Such a design method is applicable to the case of low noise levels. In general, studies of the optimal control with variances proportional to a small parameter ε have shed some light on the question of approximations [30], [57].

For larger noise levels it has been pointed out [52] that Taylor series expansions about mean values are inappropriate since the variables will, with high probability, be outside the range in which the expansions are accurate. For this reason an approximation based on averaging nonlinear effects over a broad range is more suitable. Such is the method of equivalent linearization [15], [18].

With any method the designer needs to know that the suboptimal design will yield an actual average cost reasonably close to the optimum value. Tight inequalities are required to this effect, but hardly any are available [102], [117].

V. Miscellaneous Questions

A. Dependent Noise

The assumption that the primitive random vectors are independent is not at all essential because such independence can always be achieved by expanding the state. In the extreme case, given any joint distribution of the primitive random vectors, one forms a single vector with all these variables and takes it as the random initial state. The extra components of this initial state vector are then propagated as part of the state up to the stage where they are required. With the system reformulated in this fashion, the primitive random vectors are independent because there is only one such vector, the initial state, and the state transition and output equations are noise free.

B. Parameters

The state transition and output equations may involve parameters with unknown values. In the Bayesian approach followed in this paper there will be given prior distributions for these parameters. The difference between the primitive random variables and such parameters is that the same realization of a parameter is used at several time stages (the strongest form of dependence).

Parameters may be considered as additional state variables and their prior distribution is then incorporated in the initial state distribution. The parameter values are propagated by state transition up to the stages where they are required.

Then the determination of the conditional distribution of a state vector given available data, as considered in Section IV, will *automatically* include the updating of posterior distributions of the parameters given the data. This is not a new feature of so-called adaptive control but the application of the usual stochastic control procedures to a case in which some state transition laws are particularly simple.

Parameters in the distribution of primitive random variables can be treated in several, ultimately equivalent, ways. First, they can be considered as state variables just like other parameters, with the introduction of state dependent distributions for the noise, which can be done without difficulty. Second, they can be traded directly for parameters in the equations. For instance, the mean is an additive parameter and the variance a multiplicative one. Third, one may observe that the prior distribution of the parameters, together with the noise distributions containing the parameters, determine a compound joint distribution of the noise variables. This joint distribution will violate the independence conditions, which is just the case of Section V-A.

C. Evaluating Information

The minimum (or infimum) expected cost achievable in a problem depends upon the prevailing information pattern. Changes in information produce changes in optimal cost. This suggests the idea of measuring information by its effect upon the optimal cost, as has been proposed many times [12]. For a numerical example of such changes in costs due to information changes, see [6].

Such a measure of information is entirely dependent upon the problem at hand and is clearly not additive. The only general property that it is known to possess is that additional information, if available free of charge, can never do harm though it may well be useless. This simple monotonicity property is in sharp contrast with the elaborate results of information transmission theory. The latter deals with an essentially simpler problem, because the *transmission* of the information is considered independently of its *use*, long periods of use of a transmission channel are assumed, and delays are ignored.

Efforts to establish a new theory of information, taking optimal cost into account, have not as yet been convincing [43], [85], [114], [121].

D. Related Problems

The matters discussed in this paper, concerning stochastic control, are one aspect of the more general problem of the interplay of decisions and information. Among the other aspects are worst case design, two- and n-person games, and stochastic games, all of which can occur in a discrete time formulation similar to the one discussed here. It is by no means obvious what separation properties, if any, hold in these related problems [100].

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Liquid Crystal Matrix Displays

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Abstract—Liquid crystal cells possess many properties that make their use in flat-panel displays very attractive. Most important, liquid crystal cells do not generate light but rather modify incident light, and they consume only a small amount of electrical energy while activated.

The various electrical and optical properties of liquid crystal cells which make them useful for displays are reviewed in some detail. Several methods for speeding up the dynamic response of the cells are presented. The description of a variety of addressing schemes for activating liquid crystal cells in two-dimensional matrix displays is given. A number of these schemes have been tested in conjunction with a 2×18 element display matrix where the electrical environment of a full-size TV display has been faithfully simulated. The merits of both ac and dc excitation of cells in liquid crystal displays are discussed from the standpoint of complexity of addressing circuitry and operating life, as well as power consumption of the displays.

Introduction

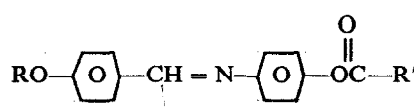
In an article that appeared in the July 1968 issue of the Proceedings of the IEEE, Heilmeier *et al.* [1] described a new electrooptic effect—dynamic scattering in nematic liquid crystals. This effect shows great promise for use in many types of displays. Since the liquid crystal does not produce light, but rather modifies incident light (either ambient or artificially supplied), it can operate at very low power. A power of 10 W/m^2 of display area is typical. Brightness equivalent to 50 percent of standard white and contrast of 20 to 1 have been achieved. The required addressing voltages are modest—approximately 40 V.

This paper discusses the application of this effect to reflective and transmissive displays. The emphasis will be on the circuit and system concepts applicable to X–Y addressed matrix displays. Properties of the liquid crystal relating to its use in such displays will also be discussed. For a more detailed and quantitative description of the physical properties of liquid crystals, the reader is referred to [1] and [2].

Dynamic Scattering Effect

Fig. 1 shows the cross section of a basic liquid crystal display element. It consists of a layer of liquid crystal material about 10 μm thick, sandwiched between two electroded glass plates. The liquid crystal has the rheological properties of a liquid, but its optical behavior is that of an ordered crystal. For nematic materials the long and rod-like molecules are in parallel alignment but are free to slide past each other. In this unperturbed state the material is transparent and the viewer sees the dark background while the incident light is reflected to its source. It is important to note that the molecules are polar, but that the polar axis is angularly displaced from the molecular axis. Since many commonly known liquid crystal materials revert to their crystalline state near room temperature, it may be necessary to operate the liquid crystal display in an environment of controlled temperature to prevent crystallization.

All of the nematic liquid crystal materials used in the test and display circuits described in this paper consist of mixtures of compounds belonging to the class of Schiff bases which have the structure



where R and R′ are n-alkyl groups [3]. These compounds characteristically have a nematic range of 25–105°C, a relative dielectric constant of approximately 3, and resistivities in the range 10^7–10^{11} $\Omega \cdot$cm, depending on the temperature and the exact composition of the material.

Fig. 2 outlines in simplified form the response of a nematic liquid crystal cell to an applied electric field. The model is consistent with the more detailed description given by Heilmeier [1], [2] and has proven quite adequate for knowledgeable exploitation of the dynamic scattering property of liquid crystals in display applications. Fig. 2(a) is the relaxed state prior to applying the voltage. Fig. 2(b) shows the element just after the voltage has been applied. The dipoles have rotated into alignment with the electric field. Continued application of the field results in generation of ions near the cathode surface which are then attracted towards the anode. The shear induced in the liquid crystal by these ions in transit creates turbulence since it tends to align the molecules in a direction parallel to the movement of the ions as shown in Fig. 2(c). Turbulence gives rise to localized variations in the index of refraction of the liquid crystal, and as a result the conditions necessary for scattering of light exist. The liquid crystal is now in a state that is called the dynamic

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